Darmstadt - Frankfurt Seminar

The Noether-Lefschetz conjecture

Wintersemester 2018

Organizers: Jan Bruinier, Yingkun Li, Torsten Wedhorn.

The Noether-Lefschetz conjecture roughly says that the Picard grouup of $\mathcal{K}_g$, the moduli space of quasi-polarized K3 surfaces of genus g, is spanned by the classes of Noether-Lefschetz divisors (more generally, one can replace the Picard group with H^{2r} of $\mathcal{K}_g$ and NL-divisors with codimension-r NL loci for $r \geq 1$). For small genus ($g \leq 12$), there is a geometric proof of this conjecture [11]. Using automorphic method, [3] gave a proof for all $g \geq 2$ and $r \leq 4$. The connection to automorphic forms come from identifying $\mathcal{K}_g$ with the Shimura variety X_K attached to the group $\mathrm{O}(19, 2)$, and the NL loci with special cycles there. The cohomology of X_K then naturally comes from automorphic forms. Using Arthur's work [2], it is now possible to classify the automorphic forms that contribute. It turns out that when r is small (compare to the dimension of the variety), all the automorphic forms are theta lifts of automorphic forms on symplectic groups of smaller ranks. These in turn come from the Kudla-Millson theta lifts, which are Poincaré duals to special cycles.

The goal of this seminar is to study and understand these ingredients that go into the proof of the Noether-Lefschetz conjecture (and generalizations). The main references will be the papers [5] (in the compact case) and [3] (in the non-compact case). There will be six meetings, each with two speakers. The speakers can freely decide the distribution and addition/subtraction of the topics, but should try to reach the goal at the end of each meeting.

In Frankfurt the seminar takes place at 15:00 in room 711 (Groß), at Robert–Mayer–Straße 10.
In Darmstadt, it takes place at 15:20 in various rooms.

1 Introduction and Statement.

Date: 01.11.2018 Darmstadt.
Speakers: Paul Kiefer, Riccardo Zuffetti.

Topics: quasi-polarized K3 surfaces, the moduli space $\mathcal{K}_g$, the Picard group of $\mathcal{K}_g$, the Noether-Lefschetz loci; orthogonal Shimura varieties and their special cycles (e.g. Heegner divisors); the relationship between these and modular forms [15, sections 4.3-4.4 and lemma 3]. Basics of Lie algebra (Cartan, Levi, Iwasawa decompositions, roots, weight of representations, Harish-Chandra homomorphism, $(\mathfrak{g}, K)$-module, admissible representation, infinitesimal character, etc), give examples for Lie algebras of classical groups such as GL_n and $\mathrm{SO}(p, q)$ (see e.g. [10, section 4.6]);

Goal: state the Noether-Lefschetz conjecture ([15, Conjecture 3]) and deduce it from Theorem 1.10 of [3].
Literature: [3, section 1], [5, sections 8.1-8.3, 9.1-9.3, 12.1-12.2], [8, sections 9, 10], [15, sections 1, 4].

2 Lie Algebra and Automorphic Forms.

Date: 15.11.2018 Frankfurt.
Speakers: Adrian Zorbach, Jennifer Kupka.

Topics: definition of Lie algebra cohomology [6, Chapter I], give the examples for orthogonal groups of signatures $(n, 1)$ [6, Chapter VI, sections 4.1-4.2] and $(n, 2)$ [5, sections 5.10]; automorphic forms on GL_2 (classical modular form, adelization to an automorphic form, Eisenstein series, Hecke operators, L-function, etc.) (see e.g. [14, sections 1, 2, 4]).

Goal: become comfortable with Lie algebra and automorphic forms.
Literature: [6], [7, sections 2.2, 2.4-2.5, 3.2], [9, sections 1-2] [10, sections 3-4], [14], [16, sections 2-3].

3 Automorphic Representations.

Date: 06.12.2018 Darmstadt.
Speakers: Jolanta Marzec, Jens Hesse

Topics: automorphic representation, strong approximation, L-function, induced representation, Eisenstein series as intertwining map, spectral decomposition of the Hilbert space $L^2(G(F)\backslash G(\mathbb{A}))$, the tensor product theorem, various types of local representations, etc.; local Langlands correspondence for GL_n (see [13] and [1] for the archimedean and non-archimedean case respectively), discuss relation to local class field theory for $n = 1$.

Goal: become comfortable with automorphic representations.
Literature: [1], [7, sections 3.1, 3.3, 3.7], [12, section 2], [13].

4 Theta Lifting.

Date: 13.12.2018 Frankfurt.
Speakers: Thomas Spittler, Alex Küronya

Topics: orthogonal/symplectic reductive dual pair, Weil representation, theta function, global theta lift, discuss Prop. 2.7 in [5]; special Schwartz forms due to Kudla-Millson, and their duality with special cycles (enough for $\mathrm{O}(p, 2)$) [5, section 7].

Goal: state Theorem 7.31 and Prop. 9.8 in [5].
Literature: [3, section 6.1-6.2], [5, sections 2.1-2.5, 7, 9.4-9.8].

5 Cohomology of Arithmetic Manifold.

Date: 24.01.2019 Darmstadt.
Speakers: Nithi Rungtanapirom, Matteo Costantini.

Topics: compare Lie algebra cohomology with L^2 and de Rham cohomologies (theorems by Zucker and Borel); cohomological unitary representations and their classifications by Vogan an Zuckerman [17] (enough for $\mathrm{SO}_0(p, q)$ with $q \leq 2$); establish the relation between $(\mathfrak{g}, K)$-cohomology and Hom functor [5, Prop. 5.4]; prove Prop. 5.16 in [5].

Goal: state and prove Theorem 8.10 (assuming corollary 6.10 and Theorem 4.2) in [5].
Literature: [3, sections 4, 7, 8.5], [5, section 5, 8.4-8.8], [6].

6 Arthur's Parameters and Span of Special Cycles.

Date: 31.01.2019 Frankfurt.
Speakers: Yingkun Li, Martin Möller.

Topics: Global and local Arthur's parameters, Arthur's packets, highly non-tempered representation, state Theorem 4.2 and Prop. 16.2 (Prop. 6.9) in [5]; proof of Theorem 10.5 and 10.7 in [5] and conclude Theorem 1.9 from these; discuss the extension to include non-cuspidal part of cohomology in [3]; discuss applications to other (orthogonal, unitary) Shimura varieties.

Goal: Prove Theorem 1.9 in [5] and wrap up the seminar.
Literature: [3, sections 8.9, 8.10], [4], [5, sections 3, 4, 10.3-10.7, 12.4, 16.5-16.11]

References

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