Introduction to Gromov–Witten Theory

Exercises

Lecture 1 Exercises

1. Recall that $\overline{\mathcal{M}}_{g,n}(X,\beta)$ has a universal family $\mathcal{U}$ such that the association
$$g \mapsto g^*\mathcal{U}$$
produces a bijection between morphisms $B \to \overline{\mathcal{M}}_{g,n}(X,\beta)$ and families of genus-g, degree-β, n-pointed stable maps over B (up to isomorphism).
   (a) Formulate the notion of isomorphism of famiies carefully.
   (b) Prove that there is a bijection between points of $\overline{\mathcal{M}}_{g,n}(X,\beta)$ and genus-g, degree-β, n-pointed stable maps up to isomorphism.

2. A **trivial family** over a base scheme B is one pulled back under the morphism $B \to \bullet$.
   (a) Formulate the notion of pullback of families carefully.
   (b) What, more explicitly, does a trivial family look like?
   (c) Let $(C; x_1, \ldots, x_n; f)$ be a genus-g, degree-β, n-pointed stable map with a nontrivial automorphism. Convince yourself that this data can be used to produce a nontrivial family over a base scheme B in which every fiber is isomorphic. (**Hint**: You can do this even with $X = \bullet$ and $\beta = 0$—that is, in $\overline{\mathcal{M}}_{g,n}$.)
   (d) Given the existence of nontrivial families in which every fiber is isomorphic, prove that a scheme $\overline{\mathcal{M}}_{g,n}(X,\beta)$ cannot have a universal family $\mathcal{U}$ producing a bijection as in the previous problem. (This is why we need $\overline{\mathcal{M}}_{g,n}(X,\beta)$ to have the structure of an orbifold.)

3. Prove that the splitting property holds in the case where $\deg(\tau_D) = 1$ and the virtual fundamental class is an ordinary fundamental class. (**Hint**: In $X \times X$, the class of the diagonal is
$$\sum_{i=1}^{k} \phi_i \boxtimes \phi^i,$$
where $\{\phi_1, \ldots, \phi_k\}$ is a basis for $H^*(X)$ with Poincaré dual basis $\{\phi^1, \ldots, \phi^k\}$ and $\boxtimes$ indicates that the two classes are pulled back under the projections $X \times X \to X$.)

4. Find an example of a boundary divisor for which the morphism
$$\tau_D : \overline{\mathcal{M}}_{g_1,n_1+1}(X,\beta_1) \times_X \overline{\mathcal{M}}_{g_2,n_2+1}(X,\beta_2) \to \overline{\mathcal{M}}_{g,n}(X,\beta)$$
does not have degree 1. How should the splitting property read in this case?

5. Come up with an identification between the forgetful map

$$\tau : \overline{\mathcal{M}}_{g,n+1}(X,\beta) \to \overline{\mathcal{M}}_{g,n}(X,\beta)$$

and the universal curve

$$\pi : \mathcal{U} \to \overline{\mathcal{M}}_{g,n}(X,\beta).$$

From the perspective of τ, what is the universal morphism $f : \mathcal{U} \to X$, and what are the universal sections $\sigma_1, \ldots, \sigma_n : \overline{\mathcal{M}}_{g,n}(X,\beta) \to \mathcal{U}$?

Lecture 2 Exercises

1. Fill in the details of the proofs of the fundamental class property, the divisor equation, and the degree-zero invariants property, under the assumption that the virtual fundamental class is an ordinary fundamental class.

2. Fill in the details of the computation of $\langle H^2\ H^2\ H^2\ H^2\rangle^{\mathbb{P}^3}_{0,4,1}$.

3. Compute all of the genus-zero primary Gromov–Witten invariants of $\mathbb{P}^1$.

4. Prove that all of the genus-zero primary Gromov–Witten invariants of $\mathbb{P}^2$ are determined by the invariants
$$N_d = \langle H^2\ \cdots\ H^2\rangle^{\mathbb{P}^2}_{0,3d-1,d}.$$
These, in turn, can be computed recursively by Kontsevich's formula; try computing N_2 from N_1 along the same lines as our computation of $\langle H^2\ H^2\ H^2\ H^2\rangle^{\mathbb{P}^3}_{0,4,1}$.

5. Fix a basis $\{\phi_0, \phi_1 \ldots, \phi_k\}$ for $H^*(X)$. Then the generating function of genus-g Gromov–Witten invariants takes as input
$$t = t_0\phi_0 + t_1\phi_1 + \cdots + t_k\phi_k$$
and is defined by
$$F_g(t) = \sum_{n,\beta} \frac{q^\beta}{n!}\langle t\ t\ \cdots\ t\rangle^X_{g,n,\beta},$$
where the t_i and q are formal variables. Prove that, in the case where $X = \mathbb{P}^k$ and $\phi_i = H^i$, then
$$F_g(t) = \sum_{d,n} \frac{q^{d\ell}e^{dt_1}}{n!}\langle t'\ t'\ \cdots t'\rangle^{\mathbb{P}^k}_{0,n,d\ell},$$
where $t' = t|_{t_1=0}$ and $\ell \in H_2(\mathbb{P}^k)$ is the class of a line.

6. The **quantum product** is a product structure $*$ on $H^*(X)[[q]]$ defined as follows: for $\gamma_1, \gamma_2 \in H^*(X)$, let
$$\big(\gamma_1 * \gamma_2, \gamma_3\big) = \sum_\beta q^\beta \langle \gamma_1\ \gamma_2\ \gamma_3\rangle^X_{0,3,\beta},$$
where $(\cdot,\cdot)$ denotes the q-linear extension of the Poincaré pairing
$$(\alpha, \beta) = \int_X \alpha \cup \beta$$
on $H^*(X)$. Then, extend $*$ to $H^*(X)[[q]]$ linearly in q.

   (a) Prove that, after setting $q = 0$, the quantum product becomes the cup product on $H^*(X)$.

   (b) Fix a basis $\{\phi_0, \phi_1 \ldots, \phi_k\}$ for $H^*(X)$, and define $F_0(t)$ as in the previous exercise. Prove that the quantum product is equivalent to
$$(\phi_i * \phi_j, \phi_k) = \frac{\partial^3}{\partial t_i \partial t_j \partial t_k} F_0(t)\bigg|_{t=0}.$$

(c) Let $X = \mathbb{P}^k$. Prove that, under the identification $q^\ell = e^{t_1}$, the quantum product is equivalent to

$$\left(\phi_i * \phi_j, \phi_k\right) = \sum_n \frac{1}{n!} \langle \phi_i \; \phi_j \; \phi_k \; (t_1 H) \; \cdots \; (t_1 H) \rangle^{\mathbb{P}^k}_{0,3+n,\beta} = \frac{\partial^3}{\partial t_i \partial t_j \partial t_k} F_0(t)\bigg|_{q=1, t_0=t_2=\cdots=t_k=0}.$$

(d) Prove that the quantum product of $\mathbb{P}^k$ is given by

$$\phi_i * \phi_j = \begin{cases} \phi_{i+j} & \text{if } i+j \leq k \\ q \; \phi_{i+j-k-1} & \text{if } i+j > k. \end{cases}$$

Conclude that, as a ring, $H^*(X)[[q]]$ with the quantum product is isomorphic to

$$\mathbb{C}[H, q]/(H^{k+1} - q).$$

Lecture 3 Exercises

These exercises concern equivariant cohomology and localization. Throughout, let M be a smooth projective variety equipped with an action of an algebraic torus $\mathbb{T} = (\mathbb{C}^*)^r$. Then an **equivariant vector bundle** on M consists of an ordinary vector bundle V on M equipped with a lift of the $\mathbb{T}$-action to the total space of V that restricts to a linear isomorphism $V_x \to V_{t \cdot x}$ on the fibers.

Equivariant vector bundles have equivariant Chern classes in $H^*_{\mathbb{T}}(M)$. As a special case, when $M = \bullet$ is a point, the generators $\lambda_1, \ldots, \lambda_r$ of $H^*_{\mathbb{T}}(\bullet)$ are defined as the first Chern classes of the equivariant line bundles $\mathcal{O}_{\lambda_i}$ given by a one-dimensional vector space with $\mathbb{T}$-action

$$(t_1, \ldots, t_r) \cdot v = t_i v.$$

More generally, when M has a trivial $\mathbb{T}$-action and $\alpha = a_1\lambda_1 + \cdots + a_r\lambda_r \in \mathbb{C}[\lambda_1, \ldots, \lambda_r]$ for some $a_1, \ldots, a_r \in \mathbb{Z}$, we denote by $\mathcal{O}_\alpha$ the non-equivariantly trivial line bundle on M with fiberwise $\mathbb{T}$-action

$$(t_1, \ldots, t_r) \cdot v = t_1^{a_1} \cdots t_r^{a_r} v. \tag{1}$$

1. Suppose that M has a trivial $\mathbb{T}$-action, lifted to the fibers of V as in (1).

   (a) Convince yourself that $V = V_0 \otimes \mathcal{O}_\alpha$, where V_0 is the same vector bundle as V but with trivial $\mathbb{T}$-action. Conclude that, if $r = \text{rank}(V)$, then

   $$c_r^{\mathbb{T}}(V) = c_r(V) + c_{r-1}(V)\alpha + \cdots + c_1(V)\alpha^{r-1} + \alpha^r.$$

   (b) Use part (a) to deduce that, in the situation where M has trivial $\mathbb{T}$-action, the equivariant top Chern class is invertible in the ring $H^*_{\mathbb{T}}(M) \otimes \mathbb{C}(\lambda_1, \ldots, \lambda_r)$.

2. Let $\mathbb{T} = (\mathbb{C}^*)^{r+1}$ act on $\mathbb{P}^r$ by

   $$(t_0, \ldots, t_r) \cdot [x_0 : \cdots : x_r] = [t_0 x_0 : \cdots : t_r x_r].$$

   What are the fixed loci of this action? Use the Atiyah–Bott localization theorem to calculate

   $$\int_{\mathbb{P}^r} c_{\text{top}}^{\mathbb{T}}(T\mathbb{P}^r),$$

   where $T\mathbb{P}^r$ is the tangent bundle of $\mathbb{P}^r$ with any lift of the $\mathbb{T}$-action.

3. Let $\mathbb{T} = (\mathbb{C}^*)^{r+1}$ act on $\mathbb{P}^r$ as above, and let V be the equivariant line bundle $\mathcal{O}_{\mathbb{P}^r}(1)$ with $\mathbb{T}$-action lifted to the total space

   $$\text{Tot}(\mathcal{O}_{\mathbb{P}^r}(1)) = \frac{(\mathbb{C}^{r+1} \setminus \{0\}) \times \mathbb{C}}{\mathbb{C}^*}, \quad (x_0, \ldots, x_r, v) \sim (\lambda x_0, \ldots, \lambda x_r, \lambda v)$$

   by

   $$(t_0, \ldots, t_r) \cdot [x_0, \ldots, x_r, v] = [t_0 x_0, \ldots, t_r x_r, v].$$

   Let $H = c_1^{\mathbb{T}}(\mathcal{O}_{\mathbb{P}^r}(1))$, the **equivariant hyperplane class**.

   (a) Prove that, if $i_j : p_j \to \mathbb{P}^r$ is the inclusion of the jth coordinate point, then $i_j^* H = \lambda_j$.

(b) The normal bundle of p_j in $\mathbb{P}^r$ is

$$N_{p_j/\mathbb{P}^r} = i_j^* T\mathbb{P}^r,$$

where $T\mathbb{P}^r$ is the tangent bundle of $\mathbb{P}^r$ with $\mathbb{T}$-action given by the derivative of the $\mathbb{T}$-action on $\mathbb{P}^r$. Use local coordinates to convince yourself that

$$i_j^* c_{\text{top}}^{\mathbb{T}}(T\mathbb{P}^r) = \prod_{k \neq j} \mathcal{O}_{\lambda_j - \lambda_k}.$$

(c) Use the above two computations and the Atiyah–Bott localization theorem to calculate

$$\int_{\mathbb{P}^2} H^2 = 1.$$

4. Let $\mathbb{T} = (\mathbb{C}^*)^{r+1}$ act on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ by post-composing stable maps $f : C \to \mathbb{P}^r$ with the above action on $\mathbb{P}^r$. Prove that any stable map of the form

$$f : \mathbb{P}^1 \to \mathbb{P}^r$$

$$f([x_0 : x_1]) = [0 : \cdots : 0 : x_0^d : 0 : \cdots : 0 : x_1^d : 0 : \cdots : 0]$$

is fixed by this action.