An introduction to Beauville surfaces via uniformization

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Abstract. A Beauville surface is a complex surface arising as a quotient of a product of hyperbolic Riemann surfaces S_1 and S_2 by the free action of a finite group G, such that the subgroup G^0 of factor-preserving elements produces quotient orbifolds S_i/G^0 of genus zero with three cone points. The study of these surfaces was initiated by F. Catanese and continued by I. Bauer, F. Grunewald and himself in a number of joint articles in which they described their basic properties and addressed the most natural questions about them.

In this paper we present the foundational results of the theory of Beauville surfaces from the point of view of uniformization, that is, in terms of Fuchsian groups. We also include new results which impose restrictions on the genera of S_1 and S_2. Finally, we construct all Beauville surfaces with Beauville group $G = \mathrm{PSL}(2, p)$ whose corresponding orbifolds S_i/G^0 have branching orders $(2, 3, n)$ and (p, p, p) respectively, where $p \geq 13$ is prime and n divides $(p \pm 1)/2$.

1. Introduction

A complex surface isogenous to the product of two compact Riemann surfaces S_1, S_2 is a complex surface of the form $X = S_1 \times S_2/G$, where G is a finite group acting freely on $S_1 \times S_2$ by biholomorphic transformations. It is known that, if the genera of S_1 and S_2 are greater than or equal to two, biholomorphic transformations of $S_1 \times S_2$ either preserve or interchange the factors S_i. If all the elements of G preserve each of the factors one speaks of surfaces of unmixed type, and of mixed type otherwise. Note that the latter can only occur if $S_1 \cong S_2$, hence in that case one can write $X = S \times S/G$. Let G^0 be the group consisting of all factor-preserving elements of G. If the quotient of each of the Riemann surfaces S_i by the action of the subgroup G^0 is an orbifold S_i/G^0 of genus zero with three cone points of orders (l_i, m_i, n_i), one says that X is a Beauville surface with group G and bitype $((l_1, m_1, n_1), (l_2, m_2, n_2))$.

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These complex surfaces have received a great deal of attention ever since the appearance of F. Catanese's article [**7**], where they were first introduced, and the papers [**2**] and [**3**] by Bauer, Catanese and Grunewald, where the basic properties were established and the study of the most natural questions was initiated. The importance of these surfaces relies on the fact that, although they are surfaces of general type, they possess striking rigidity properties, found by Catanese. For instance, two Beauville surfaces with isomorphic fundamental groups are isometric (with respect to the metric induced by the product metric on its universal cover $\mathbb{H}\times\mathbb{H}$, where $\mathbb{H}$ stands for the hyperbolic plane). In this form, the result appears in our article [**20**] but it is only a manifestation of Catanese's rigidity properties. Of course, if one allows the orbifolds S_i/G^0 to have more than 3 cone points, then the corresponding complex surfaces X will no longer be rigid and their moduli spaces will have strictly positive dimension. But these shall not be considered here.

The seminal paper [**7**] is written in the language of algebraic geometry, and one of the aims of this article is to formulate the foundational results of the theory of Beauville surfaces, contained in it and in [**2**], from the point of view of uniformization theory, thus, ultimately, in the language of Fuchsian groups. We hope that this will stimulate the interest of some Riemann surface theorists in this beautiful topic.

At the risk of over-stating the obvious, most of the material presented here is originally due to Catanese and Bauer, Catanese and Grunewald, although our approach is different. The results that, to our knowledge, are new include the following.

(i) If $X = S_1 \times S_2/G$ is an unmixed Beauville surface with pair of genera $(g(S_1), g(S_2)) = (p + 1, q + 1)$, where p and q are prime numbers, then necessarily $p = q = 5$ and X is isomorphic to the complex surface originally introduced by Beauville in [**6**] and described in Example 1 below. Moreover, this is also the only Beauville surface that reaches the minimum possible pair of genera $(6, 6)$, the next pair in the lexicographic order being $(8, 49)$, which is attained by a surface with group $\mathrm{PSL}(2, 7)$. In particular there are no Beauville surfaces with pair of genera $(6, g(S_2))$ or $(7, g(S_2))$ for any $g(S_2) > 6$ (Theorem 1).

(ii) The genus of a Riemann surface S arising in the construction of mixed Beauville surfaces is odd and greater than or equal to 17 (Corollary 4).

(iii) There are exactly $\phi(n)$ unmixed Beauville surfaces with Beauville group $G = \mathrm{PSL}(2, p)$ and bitype $((2, 3, n), (p, p, p))$, for $p \geq 13$ prime and n dividing $(p \pm 1)/2$.

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2. Triangle groups and triangle $G-$coverings

The content of this section is well known. It mostly amounts to the general statement that, via uniformization, genus zero orbifolds with three cone points correspond to normal subgroups of Fuchsian triangle groups. However some explicit choices of fundamental domain and generators of our triangle groups must be made in order to view triangle $G-$covers as triples of generators of the group G. Here we follow the account given in our recent article [**20**].

Recall that a hyperbolic orbifold of genus zero with three cone points of orders l, m, n satisfying $1/l+1/m+1/n < 1$, arises as a quotient $\mathbb{H}/\Lambda$, where Λ is a Fuchsian triangle group of signature (l, m, n). We will always place coinciding orders at the beginning of the triple, so that if two of them coincide, our triple will be (l, l, n). If the integers are all different we will always consider the triple (l, m, n) such that $l < m < n$.

To construct a triangle group of signature (l, m, n) one considers a hyperbolic triangle T in the hyperbolic plane, with vertices v_0, v_1 and v_∞ and angles π/l, π/m and π/n respectively. The reflection R_i over the edge of T opposite to v_i is an anticonformal isometry of the hyperbolic plane. The group generated by these reflections acts discontinuously on $\mathbb{H}$ in such a way that T is a fundamental domain. The index-2 subgroup formed by the orientation-preserving transformations is called a triangle group of type (l, m, n). Elementary hyperbolic theory ensures that the triangle T, and hence the corresponding triangle group, are unique up to conjugation in $\mathrm{PSL}(2, \mathbb{R})$. In the rest of the paper we reserve the notation $T = T(l, m, n)$ for the triangle in the upper half-plane $\mathbb{H}$ which is the image under $M(\mathrm{w}) = \frac{i(1+\mathrm{w})}{1-\mathrm{w}}$ of the triangle depicted in Figure 1 inside the unit disc $\mathbb{D}$, i.e. the only triangle with $v_0 = 0$, $v_\infty \in \mathbb{R}^+$ and $v_1 \in \mathbb{D}^-$, the lower half-disc. The corresponding triangle group will be denoted by $\Gamma = \Gamma(l, m, n)$.

The quadrilateral consisting of the union of T and one of its reflections $R_i(T)$ (e.g. the shaded triangle in the figure) serves as a fundamental domain for $\Gamma(l, m, n)$. Thus, the quotient $\mathbb{H}/\Gamma$ is an orbifold of genus zero with three cone points $[v_0]_\Gamma$, $[v_1]_\Gamma$ and $[v_\infty]_\Gamma$ of orders l, m and n respectively, where for an arbitrary Fuchsian group Λ the notation $[v]_\Lambda$ stands for the orbit of the point $v \in \mathbb{H}$ under the action of Λ.

It is a classical fact that $\Gamma(l, m, n)$ has presentation

$$\Gamma(l, m, n) = \langle x, y, z \ : \ x^l = y^m = z^n = xyz = 1\rangle ,$$

where $x = R_1R_\infty$, $y = R_\infty R_0$ and $z = R_0R_1$ are positive rotations around v_0, v_1 and v_∞ through angles $2\pi/l$, $2\pi/m$ and $2\pi/n$ respectively. It is also classical that any other finite order element of $\Gamma(l, m, n)$ is conjugate to a power of x, y or z and that these account for all elements in Γ that fix points. In the rest of the paper we identify $\mathbb{H}/\Gamma$ with $\mathbb{P}^1$ via the unique isomorphism

$$(1) \qquad \begin{array}{rccc} \Phi : & \mathbb{H}/\Gamma & \longrightarrow & \mathbb{P}^1 \\ & [v_0]_\Gamma & \longmapsto & 0 \\ & [v_1]_\Gamma & \longmapsto & 1 \\ & [v_\infty]_\Gamma & \longmapsto & \infty \end{array}$$

Now let G be a finite group, S a compact Riemann surface and $\mathrm{Aut}(S)$ its automorphism group. By a *triangle $G-$covering (or a $G-$orbifold of genus zero)* of type (l, m, n) we will understand a Galois covering $f : S \longrightarrow \mathbb{P}^1$ ramified over 0, 1 and ∞ with orders l, m and n respectively, such that there is a monomorphism $\mathfrak{i} : G \longrightarrow \mathrm{Aut}(S)$ where $\mathfrak{i}(G)$ agrees with the covering group $\mathrm{Aut}(S, f)$ consisting of the elements $\tau \in \mathrm{Aut}(S)$ such that $f \circ \tau = f$. Note that $\mathfrak{i}$ is only determined up to composition with an element of $\mathrm{Aut}(G)$. We will write (S, f) for such a $G-$covering, and in the rest of the paper we will always suppose that it is hyperbolic, i.e. that the genus of S is $g(S) \geq 2$.

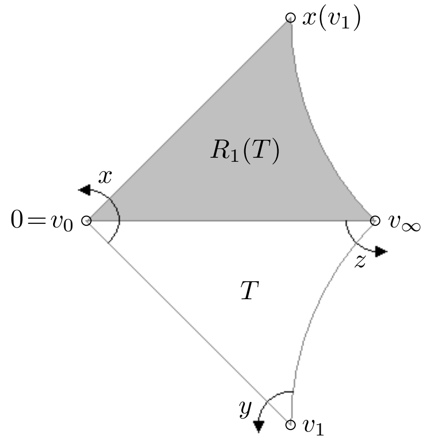

FIGURE 1. Generators x, y and z together with a fundamental domain of $\Gamma(l, m, n)$ (depicted inside the unit disc model of the hyperbolic plane).

Given (S_1, f_1) and (S_2, f_2) we say that an isomorphism $\tau : S_2 \longrightarrow S_1$ is a *strict isomorphism* of $G-$coverings if $f_2 = f_1 \circ \tau$, and we call it a *twisted isomorphism* if $f_2 = F \circ f_1 \circ \tau$ for some automorphism F of $\mathbb{P}^1$. These two concepts can be better visualized by means of the following two commutative diagrams

$$\begin{array}{ccc} S_1 & \xleftarrow{\tau} & S_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ \mathbb{P}^1 & \xrightarrow{\mathrm{Id}} & \mathbb{P}^1 \end{array} \qquad\qquad \begin{array}{ccc} S_1 & \xleftarrow{\tau} & S_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ \mathbb{P}^1 & \xrightarrow{F} & \mathbb{P}^1 \end{array}$$

Triangle $G-$coverings can be studied in a purely group theoretical way. We say that a triple (a, b, c) of elements generating G is a *hyperbolic triple of generators* of G of type (l, m, n) if the following conditions hold:

(i) $abc = 1$;
(ii) $\mathrm{ord}(a) = l$, $\mathrm{ord}(b) = m$ and $\mathrm{ord}(c) = n$;
(iii) $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$.

To such a hyperbolic triple of generators we can associate a triangle $G-$covering of type (l, m, n) in the following way. The kernel K of the epimorphism

$$\begin{array}{cccc} \rho : & \Gamma(l, m, n) & \longrightarrow & G \\ & x & \longmapsto & a \\ & y & \longmapsto & b \\ & z & \longmapsto & c \end{array} \tag{2}$$

is a torsion-free Fuchsian group so that $S = \mathbb{H}/K$ is a compact Riemann surface which carries a monomorphism $\mathfrak{i} : G \longrightarrow \mathrm{Aut}(S)$ given by the rule

$$\mathfrak{i}(g)([\mathrm{w}]_K) = [\delta(\mathrm{w})]_K, \text{ for any choice of } \delta \in \Gamma \text{ such that } \rho(\delta) = g.$$

It follows that the natural projection $\pi : \mathbb{H}/K \longrightarrow \mathbb{H}/\Gamma$ induces a triangle $G-$covering (S, f) of type (l, m, n) defined by the commutative diagram

$$
\begin{array}{ccc}
S = \mathbb{H}/K & & \\
\big\downarrow & \searrow^{f} & \\
\mathbb{H}/\Gamma & \xrightarrow{\ \Phi\ } & \mathbb{P}^1
\end{array}
\tag{3}
$$

The Riemann surface S is hyperbolic precisely because the orders l, m and n satisfy condition (iii) above, as by the Riemann–Hurwitz formula the genus $g(S)$ of S is given by

$$2g(S) - 2 = |G| \left(1 - \left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n}\right)\right). \tag{4}$$

Consider the action of $\mathrm{Aut}(G)$ on triples given by $\psi(a, b, c) := (\psi(a), \psi(b), \psi(c))$ for $\psi \in \mathrm{Aut}(G)$. Clearly the triples (a, b, c) and $\psi(a, b, c)$ give rise to the same $G-$cover.

Conversely a hyperbolic triangle $G-$covering (S, f) of type (l, m, n) determines a triple of generators of G, defined up to an element of $\mathrm{Aut}(G)$, in the following manner. Uniformization theory tells us that there is a torsion-free Fuchsian group K_1 uniformizing S, whose normalizer $N(K_1)$ contains $\Gamma = \Gamma(l, m, n)$, and an isomorphism of coverings of the form

$$
\begin{array}{ccc}
\mathbb{H}/K_1 & \xrightarrow{\ \widetilde{u}\ } & S \\
\big\downarrow & & \big\downarrow f \\
\mathbb{H}/\Gamma & \xrightarrow{\ u\ } & \mathbb{P}^1
\end{array}
$$

If the orders l, m and n are all distinct then necessarily u agrees with the isomorphism Φ defined in (1). Otherwise note that any element of $N(\Gamma)$ induces an automorphism of $\mathbb{H}/\Gamma$ which permutes the points $[v_0]_\Gamma$, $[v_1]_\Gamma$ and $[v_\infty]_\Gamma$ with equal orders. Therefore there is an element $\alpha \in N(\Gamma)$ producing the following commutative diagram

$$
\begin{array}{ccccc}
\mathbb{H}/\alpha^{-1}K_1\alpha & \xrightarrow{\ \alpha\ } & \mathbb{H}/K_1 & \xrightarrow{\ \widetilde{u}\ } & S \\
\big\downarrow & & \big\downarrow & & \big\downarrow f \\
\mathbb{H}/\Gamma & \xrightarrow{\ \alpha\ } & \mathbb{H}/\Gamma & \xrightarrow{\ u\ } & \mathbb{P}^1
\end{array}
\tag{5}
$$

where $u \circ \alpha$ equals Φ. Thus, replacing $\widetilde{\Phi}$ with $\widetilde{u} \circ \alpha$ and $\alpha^{-1}K_1\alpha$ with K, one always has a diagram of the form

$$
\begin{array}{ccc}
\mathbb{H}/K & \xrightarrow{\ \widetilde{\Phi}\ } & S \\
\big\downarrow & & \big\downarrow f \\
\mathbb{H}/\Gamma & \xrightarrow{\ \Phi\ } & \mathbb{P}^1
\end{array}
\tag{6}
$$

This yields an epimorphism $\rho : \Gamma \longrightarrow G$ (which is defined only up to an automorphism of G, just as the monomorphism $\mathfrak{i}$ is) determined by the identity

$$\widetilde{\Phi}([\gamma(\mathrm{w})]) = \mathfrak{i}\,(\rho(\gamma))\,\widetilde{\Phi}([\mathrm{w}]) \tag{7}$$

for all $\gamma \in \Gamma$, and hence a hyperbolic triple of generators

$$(a, b, c) := (\rho(x), \rho(y), \rho(z)).$$

2.1. Strict equivalence of triangle G-coverings. If in the above discussion, we start with a triangle $G-$covering (S', f') strictly isomorphic to (S, f) by means of a strict isomorphism $\tau : (S', f') \longrightarrow (S, f)$ and choose corresponding Fuchsian group representations we get a diagram as follows

$$\begin{array}{ccccccc} S & = & \mathbb{H}/K & \xleftarrow{\tau} & \mathbb{H}/K' & = & S' \\ & & f\downarrow & & \downarrow f' & & \\ \mathbb{P}^1 & = & \mathbb{H}/\Gamma & \xrightarrow{\text{Id}} & \mathbb{H}/\Gamma & = & \mathbb{P}^1 \end{array}$$

We observe that, in order for this diagram to be commutative, the isomorphism $\tau^{-1} : \mathbb{H}/K \longrightarrow \mathbb{H}/K'$ must be induced by an element $\delta \in \Gamma$. We see that the isomorphism $\widetilde{\Phi'} : \mathbb{H}/K' \longrightarrow S'$ defining the diagram analogous to (6) for the pair (S', f') is given by $\widetilde{\Phi'} = \tau^{-1} \circ \widetilde{\Phi} \circ \delta^{-1}$. Plugging this expression in the corresponding formula (7), which now reads $\widetilde{\Phi'}([\gamma(\mathrm{w})]) = \mathfrak{i}' (\rho'(\gamma)) \widetilde{\Phi'}([\mathrm{w}])$, we get the identity

$$\tau^{-1} \circ \mathfrak{i} \left(\rho(\delta^{-1}\gamma)\right) = \mathfrak{i}' \left(\rho'(\gamma)\right) \circ \tau^{-1} \circ \mathfrak{i} \left(\rho(\delta^{-1})\right).$$

It follows that $\rho' = \psi \circ \rho$, where $\psi \in \mathrm{Aut}(G)$ is defined by $\psi(g) = (\mathfrak{i}')^{-1}(g_0 \cdot \mathfrak{i}(g) \cdot g_0^{-1})$ with $g_0 = \tau^{-1} \circ \mathfrak{i} \left(\rho(\delta^{-1})\right)$. As a consequence $(a', b', c') = \psi(a, b, c)$ and we have the following proposition.

Proposition 1. *There is a bijection*

$$\left\{ \begin{array}{c} \textit{Strict isomorphism classes} \\ \textit{of triangle } G\textit{-covers } (S, f) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{Hyp. triples of generators} \\ \textit{of } G \textit{ modulo } \mathrm{Aut}(G) \end{array} \right\}$$

2.2. Twisted equivalence of triangle G-coverings. In order to prove the analogous result of Proposition 1 for twisted coverings we need to identify triples of generators modulo the action of a larger group.

It is a well-known fact (see [**26**]) that the normalizer $N(\Gamma)$ in $\mathrm{PSL}(2, \mathbb{R})$ of a triangle group $\Gamma \equiv \Gamma(l, m, n)$ is a triangle group again, and that the quotient $N(\Gamma)/\Gamma$ is faithfully represented in the symmetric group $\mathfrak{S}_3$ via its action on the vertices $[v_0]$, $[v_1]$, $[v_\infty]$ of the orbifold $\mathbb{H}/\Gamma$. Thus

$$N(\Gamma)/\Gamma \cong \begin{cases} \{1\}, & \text{if } l,\ m \text{ and } n \text{ are all distinct;} \\ \mathfrak{S}_2, & \text{if } l = m \neq n; \\ \mathfrak{S}_3, & \text{if } l = m = n. \end{cases} \tag{8}$$

where $\mathfrak{S}_k$ stands for the symmetric group on k elements.

In the second case, a representative for the non-trivial element $(1, 2) \in \mathfrak{S}_2$ is the rotation $\lambda_4 \in N(\Gamma)$ of order two around the midpoint of the segment joining v_0 and v_1 (see Figure 2). Conjugation by this element yields an order two automorphism of Γ which interchanges x and y and sends z to $x^{-1}zx$. We will denote it by $\tilde{\sigma}_4$.

In the case when $l = m = n$ we can choose the same representative λ_4 for the element $(1, 2) \in \mathfrak{S}_3$, and the order three rotation λ_1 in the positive sense around the incentre of $T(l, m, n)$ (i.e. the point where the three angle bisectors meet, see [**5**] §7.14) for $(1, 2, 3) \in \mathfrak{S}_3$. Conjugation by the latter induces an automorphism $\tilde{\sigma}_1$ of Γ of order three which sends x to y and y to z (see Figure 2).

In the following table a representative λ_i, $i = 0, \ldots, 5$, is chosen for each element of $\mathfrak{S}_3 \cong N(\Gamma)/\Gamma$, and for each automorphism $\tilde{\sigma}_i$ of Γ obtained by conjugation by λ_i, its action on the triple of generators x, y, z is indicated. The table describes the case in which $l = m = n$, but the other two cases are also contained in it, for

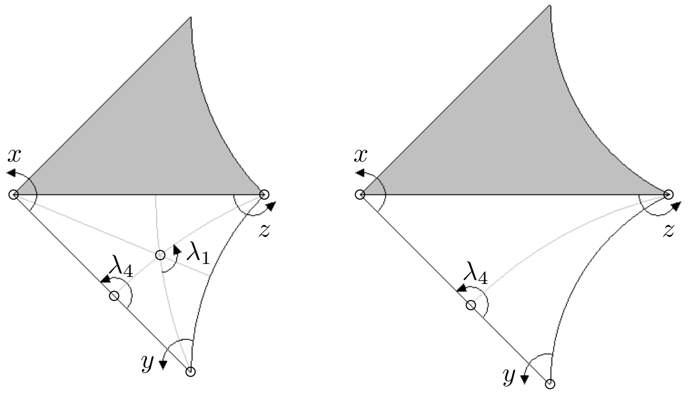

FIGURE 2. Generators of $\Gamma(l,l,l)$ and $\Gamma(l,l,n)$, and representatives of $(1,2),(1,2,3) \in \mathfrak{S}_3$.

obviously the case $l = m \neq n$ corresponds to the first and the fifth lines, and the case where l, m, n are all different corresponds to just the identity.

Permutation	Representatives of $N(\Gamma)/\Gamma$	$\mathrm{Aut}(\Gamma)$	Action on the generators of Γ
Id	$\lambda_0 = \mathrm{Id}$	$\tilde{\sigma}_0 \equiv \mathrm{Id}$	(x,y,z)
$(1,2,3)$	λ_1	$\tilde{\sigma}_1 : \ \gamma \mapsto \lambda_1\gamma\lambda_1^{-1}$	(y,z,x)
$(1,3,2)$	$\lambda_2 = \lambda_1^2$	$\tilde{\sigma}_2 : \ \gamma \mapsto \lambda_2\gamma\lambda_2^{-1}$	(z,x,y)
$(1,3)$	$\lambda_3 = \lambda_1\lambda_4$	$\tilde{\sigma}_3 : \ \gamma \mapsto \lambda_3\gamma\lambda_3^{-1}$	$(z,y,y^{-1}xy)$
$(1,2)$	λ_4	$\tilde{\sigma}_4 : \ \gamma \mapsto \lambda_4\gamma\lambda_4^{-1}$	$(y,x,x^{-1}zx)$
$(2,3)$	$\lambda_5 = \lambda_1^2\lambda_4$	$\tilde{\sigma}_5 : \ \gamma \mapsto \lambda_5\gamma\lambda_5^{-1}$	$(x,z,z^{-1}yz)$

TABLE 1. Correspondence $N(\Gamma)/\Gamma \cong \mathfrak{S}_3$.

It is worth noting that in the case when $N(\Gamma)/\Gamma = \mathfrak{S}_2$ or $\{1\}$ the extension splits, but when $N(\Gamma)/\Gamma = \mathfrak{S}_3$ it does not, since no Fuchsian group can contain a noncyclic finite group. This means that the representatives of $N(\Gamma)/\Gamma$ cannot be chosen naturally to form a complement of Γ.

To summarize, $N(\Gamma)$ can be written as

$$N(\Gamma) \cong \begin{cases} \Gamma, & \text{if } l,\ m \text{ and } n \text{ are all distinct;} \\ \langle\, \Gamma, \lambda_4 \,\rangle, & \text{if } l = m \neq n; \\ \langle\, \Gamma, \lambda_1, \lambda_4 \,\rangle, & \text{if } l = m = n. \end{cases} \tag{9}$$

Given a finite group G, we introduce for convenience the following bijections of the set $\mathbb{T}(G; l, m, n)$ of hyperbolic triples of generators of G of a given type (l, m, n). They are defined in the following way

$$\begin{array}{ll} \sigma_0(a,b,c) = (a,b,c) & \sigma_3(a,b,c) = (c,b,b^{-1}ab) \\ \sigma_1(a,b,c) = (b,c,a) & \sigma_4(a,b,c) = (b,a,a^{-1}ca) \\ \sigma_2(a,b,c) = (c,a,b) & \sigma_5(a,b,c) = (a,c,c^{-1}bc). \end{array} \tag{10}$$

Note that they are defined so as to satisfy

$$\begin{aligned}(11)\qquad \sigma_i(\rho(x),\rho(y),\rho(z)) &= \left(\rho\left(\lambda_i x\lambda_i^{-1}\right),\rho\left(\lambda_i y\lambda_i^{-1}\right),\rho\left(\lambda_i z\lambda_i^{-1}\right)\right) = \\ &= \left(\rho\left(\tilde{\sigma}_i(x)\right),\rho\left(\tilde{\sigma}_i(y)\right),\rho\left(\tilde{\sigma}_i(z)\right)\right),\end{aligned}$$

where $\rho:\Gamma(l,m,n)\longrightarrow G$ is the epimorphism associated in (2) to each triple of generators.

Remark 1. We follow here Bauer, Catanese and Grunewald's notation in [**2**], although there is a discrepancy in the definition of σ_3 and σ_4 due to the choice of different representatives for the classes of λ_3 and λ_4 in $N(\Gamma)/\Gamma$.

In order to understand the relationship between triples of generators of G and twisted isomorphism classes of triangle $G-$coverings, we will need to consider the following group of bijections of $\mathbb{T}(G;l,m,n)$

$$\mathrm{A}(G;l,m,n)=\begin{cases}\mathrm{Aut}(G), & \text{if } l,\ m,\ n \text{ are all distinct;}\\ \langle\,\mathrm{Aut}(G),\sigma_4\,\rangle, & \text{if } l=m\neq n;\\ \langle\,\mathrm{Aut}(G),\sigma_1,\dots,\sigma_5\,\rangle, & \text{if } l=m=n.\end{cases}$$

The action of the composition of two elements σ_i and σ_j on a triple (a,b,c) follows the following table

	$\boldsymbol{\sigma_0}$	$\boldsymbol{\sigma_1}$	$\boldsymbol{\sigma_2}$	$\boldsymbol{\sigma_3}$	$\boldsymbol{\sigma_4}$	$\boldsymbol{\sigma_5}$
$\boldsymbol{\sigma_0}$	σ_0	σ_1	σ_2	σ_3	σ_4	σ_5
$\boldsymbol{\sigma_1}$	σ_1	σ_2	σ_0	$\gamma_{b^{-1}}\circ\sigma_4$	$\gamma_{a^{-1}}\circ\sigma_5$	$\gamma_{c^{-1}}\circ\sigma_3$
$\boldsymbol{\sigma_2}$	σ_2	σ_0	σ_1	$\gamma_c\circ\sigma_5$	$\gamma_b\circ\sigma_3$	$\gamma_a\circ\sigma_4$
$\boldsymbol{\sigma_3}$	σ_3	σ_5	σ_4	$\gamma_{b^{-1}}\circ\sigma_0$	$\gamma_{a^{-1}}\circ\sigma_2$	$\gamma_{c^{-1}}\circ\sigma_1$
$\boldsymbol{\sigma_4}$	σ_4	σ_3	σ_5	σ_1	σ_0	σ_2
$\boldsymbol{\sigma_5}$	σ_5	σ_4	σ_3	$\gamma_c\circ\sigma_2$	$\gamma_b\circ\sigma_1$	$\gamma_a\circ\sigma_0$

where the product $\sigma_i\cdot\sigma_j$ is to be found in the intersection of the $i-$th row and the $j-$th column, and γ_g stands for conjugation by an element $g\in G$. Using this table, one can easily check that G is normal in $\mathrm{A}(G;l,m,n)$.

As a consequence the action of any element $\mu\in\mathrm{A}(G;l,m,n)$ on a specific triple (a,b,c) can be written as $\mu=\psi\circ\sigma_i$ for some σ_i, $i=0,\dots,5$, where ψ is an automorphism of G. We note that in general ψ depends on the triple (a,b,c).

Given an element $\delta\in\mathrm{PSL}(2,\mathbb{R})$, we will write φ_δ for conjugation by δ.

Lemma 1. *The following two statements are equivalent:*

(i) $(a,b,c)\equiv(a',b',c')\bmod \mathrm{A}(G;l,m,n)$*;*
(ii) *there exist* $\psi\in\mathrm{Aut}(G)$ *and* $\delta\in N(\Gamma)$ *such that* $\rho'=\psi\circ\rho\circ\varphi_\delta$*.*

Proof. Let us suppose that $(a,b,c)\equiv(a',b',c')\bmod \mathrm{A}(G;l,m,n)$. By the comments above there exists a transformation σ_i such that $(a',b',c')=\psi(\sigma_i(a,b,c))$. Therefore, using (11), we have

$$(\rho'(x),\rho'(y),\rho'(z))=(a',b',c')=\psi(\rho(\tilde{\sigma}_i(x)),\rho(\tilde{\sigma}_i(y)),\rho(\tilde{\sigma}_i(z))).$$

For the converse, note that by (9) every $\delta\in N(\Gamma)$ is of the form $\delta=\eta\lambda_i$, for some λ_i, $i=0,\dots,5$ and $\eta\in\Gamma$.

Therefore, we can write

$$(a',b',c')=\psi\left(\rho(\varphi_\delta(x)),\rho(\varphi_\delta(y)),\rho(\varphi_\delta(z))\right)=\psi(g(\sigma_i(a,b,c))g^{-1}),$$

where $g=\rho(\eta)$. □

For later use we record the following remark.

Remark 2. If instead of the group $\mathrm{A}(G;l,m,n)$ we restrict ourselves to the subgroup

$$\mathrm{I}(G;l,m,n)=\begin{cases} G, & \text{if } l,\ m,\ n \text{ are all distinct;}\\ \langle\, G,\sigma_4\,\rangle, & \text{if } l=m\neq n;\\ \langle\, G,\sigma_1,\sigma_4\,\rangle, & \text{if } l=m=n. \end{cases}$$

where G acts on $\mathbb{T}(G;l,m,n)$ by conjugation, then the corresponding result in Lemma 1 will be that $(a,b,c)\equiv(a',b',c') \bmod \mathrm{I}(G;l,m,n)$ if and only if $\rho'=\rho\circ\varphi_\delta$ for some $\delta\in N(\Gamma)$.

More precisely, if $(a',b',c')=g\cdot(\sigma_i(a,b,c))\cdot g^{-1}$, then the element $\delta\in N(\Gamma)$ can be taken to be $\delta=\eta\lambda_i$, for any $\eta\in\Gamma$ such that $g=\rho(\eta)$.

We can now prove the analogue of Proposition 1 for the twisted case, namely

Proposition 2. *There is a bijection*

$$\left\{\begin{array}{c}\textit{Twisted isomorphism classes}\\ \textit{of triangle } G\textit{-covers } (S,f)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\textit{Hyp. triples of generators}\\ \textit{of } G \textit{ modulo } \mathrm{A}(G;l,m,n)\end{array}\right\}$$

Proof. Let (a,b,c), (a',b',c') be two triples of hyperbolic generators of G determining two epimorphisms ρ and ρ', and hence two triangle G−coverings as in (3). If $(a',b',c')\equiv(a,b,c) \bmod \mathrm{A}(G;l,m,n)$ then by Lemma 1 one has the equality $K':=\ker\rho'=\delta K\delta^{-1}$ and a commutative diagram as follows

$$\begin{array}{ccccccc} S & = & \mathbb{H}/K & \xrightarrow{\ \delta\ } & \mathbb{H}/K' & = & S' \\ & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} & & \\ & & \mathbb{P}^1 & \xrightarrow{\ F\ } & \mathbb{P}^1 & & \end{array}$$

where $F=\Phi\circ\overline{\delta}\circ\Phi^{-1}$ and $\overline{\delta}$ is the automorphism of $\mathbb{H}/\Gamma$ induced by δ. Therefore, in this case, the corresponding coverings (S,f) and (S',f') are twisted isomorphic.

Conversely, if we start with a twisted isomorphism of coverings τ between (S,f) and (S',f'), then there is a commutative diagram of the form

$$\begin{array}{ccccc} S & \xrightarrow{\ \mathrm{Id}\ } & S & \xrightarrow{\ \tau\ } & S' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f'} \\ \mathbb{P}^1 & \xrightarrow{\ F\ } & \mathbb{P}^1 & \xrightarrow{\ \mathrm{Id}\ } & \mathbb{P}^1 \end{array}$$

where $(S,f_1):=(S,F\circ f)$ for a suitable Möbius transformation F. Since (S,f_1) and (S',f') are strictly isomorphic, there is an automorphism $\psi\in\mathrm{Aut}(G)$ such that their corresponding epimorphisms ρ_1 and ρ' are related by $\rho_1=\psi\circ\rho'$. Now, as explained in the previous sections (see (5) and (6)), from the Fuchsian group point of view the coverings (S,f) and (S,f_1) correspond to diagrams

$$\begin{array}{ccc} \mathbb{H}/K & \xrightarrow{\ \widetilde{\Phi}\ } & S \\ \downarrow & & \downarrow{\scriptstyle f} \\ \mathbb{H}/\Gamma & \xrightarrow{\ \Phi\ } & \mathbb{P}^1 \end{array} \qquad\qquad \begin{array}{ccc} \mathbb{H}/\delta^{-1}K\delta & \xrightarrow{\ \widetilde{\Phi}_1\ } & S \\ \downarrow & & \downarrow{\scriptstyle F\circ f} \\ \mathbb{H}/\Gamma & \xrightarrow{\ \Phi\ } & \mathbb{P}^1 \end{array}$$

where $\widetilde{\Phi}_1=\widetilde{\Phi}\circ\delta$ and $\delta\in N(\Gamma)$ induces the automorphism $\overline{\delta}:\mathbb{H}/\Gamma\longrightarrow\mathbb{H}/\Gamma$ such that $F\circ\Phi\circ\overline{\delta}=\Phi$. As a consequence the epimorphism ρ_1 corresponding to $(S,F\circ f)$

is defined by the equality

$$\widetilde{\Phi}_1([\gamma(\mathrm{w})]) = \mathfrak{i}\,(\rho_1(\gamma))\,\widetilde{\Phi}_1([\mathrm{w}])\,,$$

and therefore $\rho_1(\gamma) = \rho(\delta\gamma\delta^{-1})$.

By Lemma 1, since $\rho(\gamma) = \rho_1(\delta^{-1}\gamma\delta) = \psi \circ \rho'(\delta^{-1}\gamma\delta)$, we finally have that $(a, b, c) \equiv (a', b', c') \bmod \mathrm{A}(G; l, m, n)$. □

2.3. The complex conjugate orbifold. A Riemann surface S is said to be *real* if it admits an anticonformal involution, i.e. if there exists an anticonformal isomorphism $h : S \longrightarrow S$ such that $h^2 \equiv \mathrm{Id}$. The term real comes from the fact that such a Riemann surface is known to be isomorphic to the Riemann surface S_F corresponding to an algebraic curve $F(x, y) = 0$ with real coefficients. Note that the converse obviously holds, for if the polynomial $F(x, y)$ has real coefficients, the map $(x, y) \mapsto (\overline{x}, \overline{y})$ induces the required anticonformal involution.

Now, given a $G-$orbifold (S, f) we can construct the complex conjugate orbifold $(\overline{S}, \overline{f})$, where $\overline{S}$ is the complex conjugate Riemann surface of S and the covering $\overline{f}$ is defined by $\overline{f}(P) = \overline{f(P)}$. Remember that if S is given by an atlas $\{(U_i, \varphi_i)\}$, then $\overline{S}$ is obtained simply by considering the complex conjugate atlas $\{(U_i, \overline{\varphi_i})\}$.

Note that the function $\overline{f}$ is locally given by $z \mapsto \overline{f} \circ \overline{\varphi_i}^{-1}(z) = \overline{f \circ \varphi_i^{-1}(\overline{z})}$, hence it is holomorphic. A similar argument proves that the holomorphic homeomorphisms of S coincide with the holomorphic homeomorphisms of $\overline{S}$.

It follows that the cover $(\overline{S}, \overline{f})$ comes automatically equipped with a group isomorphism $G \longrightarrow \mathrm{Aut}(\overline{S}, \overline{f}) = \mathrm{Aut}(S, f)$, which coincides with the isomorphism induced by $\mathfrak{i}$.

As an application of Proposition 1, in this section we prove the following fact.

Proposition 3. *Let S be a Riemann surface admitting a triangle $G-$cover (S, f) and suppose that there exists a strict isomorphism between (S, f) and $(\overline{S}, \overline{f})$. Then the Riemann surface S is real.*

This proposition should be compared with the following result by C. Earle, which shows that not all Riemann surfaces isomorphic to their complex conjugates are real.

Theorem ([**9**], [**10**]). *Let S_t be the compact Riemann surface of genus two determined by the equation $w^2 = z(z^2 - \xi)(z^2 + tz - 1)$, where $\xi = \exp(2\pi i/3)$. Then for $t > 0$ the Riemann surface S_t has an antiholomorphic automorphism of order four (hence it is isomorphic to $\overline{S_t}$) but it has no antiholomorphic involution unless $t = 1$.*

We observe that in [**10**] it is proved that the exceptional Riemann surface S_1 is real. Moreover Earle's computations show that S_1 does not satisfy the conditions of our Proposition 3. As a matter of fact, along the proof of this theorem Earle shows that the group $\mathrm{Aut}(S_1)$ is generated by the hyperelliptic involution j together with another order two automorphism τ which is the lift to S_1 of a Möbius transformation he denotes AB^3. Now, the action of τ splits the set of Weierstrass points (the six points fixed by j) into three pairs, which along with the two fixed points of AB^3 gives a quotient orbifold $S_1/\mathrm{Aut}(S_1)$ of genus zero with more than three cone points (actually, five cone points of order two).

Other examples of Riemann surfaces which are isomorphic to their complex conjugates but cannot be defined by real polynomials were published a little later by G. Shimura ([**25**]) and more recently by R. Hidalgo ([**21**]).

Proof of Proposition 3. We will work here with the unit disc $\mathbb{D}$ instead of the upper half-plane. First let us note that if

$$\begin{array}{ccc} \mathbb{D}/K & \xrightarrow{\widetilde{\Phi}} & S \\ \downarrow & & \downarrow f \\ \mathbb{D}/\Gamma & \xrightarrow{\Phi} & \mathbb{P}^1 \end{array}$$

is the commutative diagram expressing the covering (S,f) in terms of Fuchsian groups, as in (3), then the corresponding diagram for the covering $(\overline{S},\overline{f})$ is

$$\begin{array}{ccc} \mathbb{D}/\overline{K} & \xrightarrow{\widetilde{\Phi}_1} & \overline{S} \\ \downarrow & & \downarrow \overline{f} \\ \mathbb{D}/\overline{\Gamma} & \xrightarrow{\Phi_1} & \mathbb{P}^1 \end{array}$$

where $\widetilde{\Phi}_1(\mathrm{w}) = \widetilde{\Phi}(\overline{\mathrm{w}})$, $\Phi_1(\mathrm{w}) = \overline{\Phi(\overline{\mathrm{w}})} = \overline{\Phi}(\overline{\mathrm{w}})$ and for a subgroup H of $\mathrm{Aut}(\mathbb{D})$ we put $\overline{H} := \{h' : h \in H\}$, where we write h' for the element of $\mathrm{Aut}(D)$ obtained by applying complex conjugation to the coefficients of a given Möbius transformation h. Now, since $x(\mathrm{w}) = \xi_l \cdot \mathrm{w}$ and z is conjugate to $\mathrm{w} \mapsto \xi_n \cdot \mathrm{w}$ by means of a real Möbius transformation (see Figure 1) we see that $x' = x^{-1}$ and $z' = z^{-1}$. It follows that $\overline{\Gamma} = \Gamma$. Note that $\widetilde{\Phi}_1$ and Φ_1 are equivariant respect to $\overline{K}$ and $\overline{\Gamma} = \Gamma$ respectively, and that $\overline{f} \circ \widetilde{\Phi}_1(\mathrm{w}) = \Phi_1 \circ \pi(\mathrm{w})$, where π stands for the projection $\mathbb{D}/\overline{K} \longrightarrow \mathbb{D}/\overline{\Gamma}$. Moreover the function Φ_1 induces the same isomorphism $\mathbb{D}/\Gamma \simeq \mathbb{P}^1$ as Φ.

Therefore comparing the formulae (7), corresponding to the coverings (S,f) and $(\overline{S},\overline{f})$, we see that the associated epimorphisms ρ and ρ' are given by

$$\begin{array}{rccl} \rho: & \Gamma(l,m,n) & \longrightarrow & G \\ & x & \longmapsto & a \\ & y & \longmapsto & b \\ & z & \longmapsto & c \end{array} \qquad \begin{array}{rccl} \overline{\rho}: & \Gamma(l,m,n) & \longrightarrow & G \\ & x & \longmapsto & a^{-1} \\ & y & \longmapsto & ab^{-1}a^{-1} \\ & z & \longmapsto & c^{-1} \end{array}$$

In other words we have $\overline{\rho}(\gamma) = \rho(\gamma')$.

Now, since (S,f) and $(\overline{S},\overline{f})$ are strictly isomorphic coverings, Proposition 1 implies that the triples (a,b,c) and $(a^{-1},ab^{-1}a^{-1},c^{-1})$ are related by an automorphism $\psi \in \mathrm{Aut}(G)$, i.e. $\overline{\rho} = \psi \circ \rho$, and therefore $\overline{K} = \ker \overline{\rho} = K$. Hence the rule $\mathrm{w} \longmapsto \overline{\mathrm{w}}$ is an anticonformal involution of $S = \mathbb{H}/K$. □

Remark 3. It is worthwhile to point out that Proposition 3 is a particular case of a much more general result concerning not only complex conjugation of Riemann surfaces, but arbitrary Galois conjugation of algebraic curves (see [**27**]).

3. The concept of Beauville surface

We say that a complex surface X is *isogenous to a product* if it is isomorphic to the quotient of a product of Riemann surfaces $S_1 \times S_2$ of genus $g(S_1), g(S_2) \geq 1$

by the free action of a finite group $G < \mathrm{Aut}(S_1 \times S_2)$. If $g(S_1), g(S_2) \geq 2$ we say that X is *isogenous to a higher product.*

First of all, let us note (see (17) below) that each element of $\mathrm{Aut}(S_1 \times S_2)$ either fixes each Riemann surface or interchanges them. Clearly if two elements $g, h \in G$ both interchange factors, their product gh does not. In particular if we denote by $G^0 < G$ the subgroup of factor-preserving elements, then $[G : G^0] \leq 2$.

A particular case of surfaces isogenous to a product are Beauville surfaces, introduced by F. Catanese in [**7**] following a construction of A. Beauville in [**6**] (see Example 1 below).

A *Beauville surface* is a compact complex surface X satisfying the following properties:

(i) X is isogenous to a higher product, $X \cong S_1 \times S_2/G$;
(ii) Let $G^0 \lhd G$ be the subgroup of factor-preserving elements. Then G^0 acts effectively on each of the Riemann surfaces S_i producing quotient orbifolds S_i/G^0 of genus zero with three cone points.

We will say that X is of *unmixed type* (or that X is an *unmixed Beauville surface*) if $G = G^0$ and that it is of *mixed type* (or that it is a *mixed Beauville surface*) if $G \neq G^0$. Let us remark that in the mixed case necessarily $S_1 \cong S_2$.
If (l_1, m_1, n_1) and (l_2, m_2, n_2) are the types of the G^0-coverings S_1 and S_2, we will say that the Beauville surface $X = S_1 \times S_2/G$ has *bitype* $((l_1, m_1, n_1), (l_2, m_2, n_2))$.

Example 1 (Beauville)**.** Consider the Fermat curve of degree five

$$F_5 = \{x^5 + y^5 + z^5 = 0\}.$$

The group $G = (\mathbb{Z}/5\mathbb{Z})^2$ acts freely on $F_5 \times F_5$ in the following way: for each $(\alpha, \beta) \in G$ define $e_{\alpha,\beta} : F_5 \times F_5 \longrightarrow F_5 \times F_5$ as

$$\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \right) \longmapsto \left(\begin{bmatrix} \xi^{\alpha} x_1 \\ \xi^{\beta} y_1 \\ z_1 \end{bmatrix}, \begin{bmatrix} \xi^{\alpha+3\beta} x_2 \\ \xi^{2\alpha+4\beta} y_2 \\ z_2 \end{bmatrix} \right),$$

where $\xi = e^{2\pi i/5}$.

Then $X := F_5 \times F_5/(\mathbb{Z}/5\mathbb{Z})^2$ is an unmixed Beauville surface.

Beauville surfaces with abelian Beauville group have been studied and classified ([**7**], [**2**], [**16**], [**19**]). All of them arise as quotients of $F_n \times F_n$ by some action of the group $(\mathbb{Z}/n\mathbb{Z})^2$, where F_n stands for the Fermat curve

$$F_n = \{[x : y : z] \in \mathbb{P}^2(\mathbb{C}) : x^n + y^n + z^n = 0\}$$

and $\gcd(n, 6) = 1$. The number of isomorphism classes of Beauville surfaces which have Beauville group $(\mathbb{Z}/n\mathbb{Z})^2$ is given by a polynomial in n of degree 4 in the case of prime powers, and by a much more complicated formula in the general case. A consequence of these formulae is that for $n = 5$ there is only one Beauville surface with group $(\mathbb{Z}/5\mathbb{Z})^2$, namely the one above originally constructed by Beauville.

4. Uniformization of Beauville surfaces: unmixed case

Let now $X = S_1 \times S_2/G$ be a Beauville surface and let us consider first the unmixed case, i.e. the case in which $G = G^0$. Clearly its holomorphic universal cover is the bidisc $\mathbb{H} \times \mathbb{H}$ and the covering group is a subgroup of $\mathrm{Aut}(\mathbb{H} \times \mathbb{H})$. Let us denote it by Γ_{12}, so that $X = \mathbb{H}\times\mathbb{H}/\Gamma_{12}$ with $\Gamma_{12} \cong \pi_1(X)$. The first condition in the definition of Beauville surface implies that there is an exact sequence of the form

$$1 \longrightarrow K_1 \times K_2 \longrightarrow \Gamma_{12} \stackrel{\rho}{\longrightarrow} G \longrightarrow 1 \tag{12}$$

where K_1 and K_2 uniformize two compact Riemann surfaces $S_1 = \mathbb{H}/K_1$ and $S_2 = \mathbb{H}/K_2$ and the group $G \cong \Gamma_{12}/K_1 \times K_2$ acts on $S_1 \times S_2$ as follows. Let g be an element of G. If $(\gamma_1, \gamma_2) \in \Gamma_{12}$ is such that $\rho(\gamma_1, \gamma_2) = g$, then the action of g on points $[\mathrm{w}_1, \mathrm{w}_2] \in \mathbb{H}/K_1 \times \mathbb{H}/K_2$ is given by the rule

$$g([\mathrm{w}_1, \mathrm{w}_2]) = [\gamma_1(\mathrm{w}_1), \gamma_2(\mathrm{w}_2)]\,,$$

while the action of g on the individual factors is given by $g([\mathrm{w}_1]) = [\gamma_1(\mathrm{w}_1)]$ and $g([\mathrm{w}_2]) = [\gamma_2(\mathrm{w}_2)]$.

Now, by the second condition in the definition, the quotients $\Gamma_1 \cong \Gamma_{12}/K_2$ and $\Gamma_2 \cong \Gamma_{12}/K_1$ of the group Γ_{12} must be triangle groups defining triangle $G-$covers $f_i : S_i \cong \mathbb{H}/K_i \longrightarrow \mathbb{P}^1 \cong \mathbb{H}/\Gamma_i$ with $G \cong \Gamma_i/K_i$. Therefore there are two exact sequences

$$1 \longrightarrow K_i \longrightarrow \Gamma_i \stackrel{\rho_i}{\longrightarrow} G \longrightarrow 1 \qquad (i = 1, 2)$$

representing the action of G on the individual factors so that, in particular, for the element (γ_1, γ_2) above one must have $\rho_1(\gamma_1) = \rho_2(\gamma_2) = g$. It follows that

$$\Gamma_{12} = \{(\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2 \ : \ \rho_1(\gamma_1) = \rho_2(\gamma_2)\} < \Gamma_1 \times \Gamma_2\,. \tag{13}$$

Let (a_i, b_i, c_i) be a generating triple defining the $G-$cover (S_i, f_i). Then the subsets of G

$$\Sigma(a_i, b_i, c_i) := \bigcup_{g\in G} \bigcup_{j=1}^{\infty} \{g a_i^j g^{-1}, g b_i^j g^{-1}, g c_i^j g^{-1}\}\ , \quad (i = 1, 2)$$

consisting of the elements of G that fix points on S_1 and S_2 respectively, necessarily have trivial intersection, that is

$$\Sigma(a_1, b_1, c_1) \cap \Sigma(a_2, b_2, c_2) = \{1\}\,, \tag{14}$$

for otherwise the action of G on $S_1 \times S_2$ would not be free.

Conversely, any pair of hyperbolic triples of generators (a_1, b_1, c_1), (a_2, b_2, c_2) of G satisfying condition (14) define via the associated epimorphisms ρ_1, ρ_2 a group $\Gamma_{12} < \Gamma_1 \times \Gamma_2$ as in (13), which clearly uniformizes a Beauville surface.

Corollary 1 ([**7**]). *Let G be a finite group. Then there are Riemann surfaces S_1 and S_2 of genera $g(S_1), g(S_2) > 1$ and an action of G on $S_1 \times S_2$ so that $S_1 \times S_2/G$ is an unmixed Beauville surface if and only if G has two hyperbolic triples of generators (a_i, b_i, c_i) of order (l_i, m_i, n_i), $i = 1, 2$, satisfying the compatibility condition* (14).

Under these assumptions one says that such a pair of triples $(a_1, b_1, c_1; a_2, b_2, c_2)$ is an *unmixed Beauville structure* on G.

Example 2. By the last corollary, corresponding to Beauville's original surface described in Example 1 there should be a pair of triples of generators of $G =$

$(\mathbb{Z}/5\mathbb{Z})^2$ of type $(5,5,5)$ satisfying the compatibility condition above. In fact the following two triples will do

$$a_1 = (1,0), \quad b_1 = (0,1), \quad c_1 = (4,4),$$
$$a_2 = (3,1), \quad b_2 = (4,2), \quad c_2 = (3,2).$$

The compatibility condition is easily checked, and in fact it is not hard to see that the Riemann surface defined by these triples is in both cases the Fermat curve of degree five. To prove this first note that, since all the elements in both triples have order 5, the two corresponding Riemann surfaces will be uniformized by surface subgroups K_1 and K_2 of the triangle group $\Gamma = \Gamma(5,5,5)$. As the quotient $\Gamma/K_i = G$ is abelian, the groups K_i must contain the commutator $[\Gamma,\Gamma]$. But $\Gamma/[\Gamma,\Gamma]$ is already isomorphic to $(\mathbb{Z}/5\mathbb{Z})^2$, so $K_1 = K_2 = [\Gamma,\Gamma]$, and this group is known to uniformize the Fermat curve of degree 5 (see e.g. [**14**], [**18**]).

4.1. Some restrictions to the existence of unmixed Beauville surfaces. A natural problem regarding Beauville surfaces $X = S_1 \times S_2/G$ is to determine which genera $g(S_1)$ of S_1 and $g(S_2)$ of S_2 can arise in their construction. In [**14**] it was shown that $g(S_1), g(S_2) \geq 6$. In this section we improve that result.

Perhaps the most direct way to get restrictions on the genera $g(S_1)$ and $g(S_2)$ is to combine Riemann–Hurwitz's formula (4) with the formula giving the Euler–Poincaré characteristic of X, namely

$$\chi(X) = \frac{\chi(S_1)\cdot\chi(S_2)}{|G|} = \frac{(2g(S_1)-2)(2g(S_2)-2)}{|G|}, \tag{15}$$

the relevant fact being that this fraction has to be a natural number.

Actually an even stronger ingredient is obtained by considering the holomorphic Euler characteristic of X, defined as the alternating sum of the dimensions of the cohomology groups of the structural sheaf, i.e. $\chi(\mathcal{O}_X) = h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) + h^2(\mathcal{O}_X)$ (see e.g. [**6**] or [**1**]). In the case of a surface isogenous to a product we have

$$\chi(\mathcal{O}_X) = \frac{(g(S_1)-1)(g(S_2)-1)}{|G|}, \qquad \text{i.e.} \quad \chi(\mathcal{O}_X) = \frac{\chi(X)}{4} \tag{16}$$

and the point is, of course, that this fraction is still a natural number.

The last identity follows from Noether's formula, a central result of the theory of complex surfaces, which states that

$$\chi(\mathcal{O}_X) = \frac{1}{12}(K_X^2 + \chi(X)).$$

Here, as usual, K_Y^2 denotes the degree of the self-intersection of the canonical class of a complex surface Y. In the particular case in which $Y = S_1 \times S_2$, the degree K_Y^2 can be computed by considering generic holomorphic 1−forms ω_1, ω_1' of S_1 and ω_2, ω_2' of S_2 and looking at the intersection of $\mathcal{Z}(\eta_1)$ and $\mathcal{Z}(\eta_2)$, the zero sets of the 2−forms $\eta_1 = \omega_1 \wedge \omega_2$ and $\eta_2 = \omega_1' \wedge \omega_2'$. Denoting intersection by $\cdot$ and union by $+$, as it is customary in intersection theory, we have

$$\mathcal{Z}(\eta_1)\cdot\mathcal{Z}(\eta_2) = \big((\mathcal{Z}(\omega_1)\times S_2) + (S_1 \times \mathcal{Z}(\omega_2))\big)\cdot\big((\mathcal{Z}(\omega_1')\times S_2) + (S_1\times\mathcal{Z}(\omega_2'))\big)$$

which by the Riemann–Roch theorem for Riemann surfaces is a set consisting of $2(2g(S_1)-2)(2g(S_2)-2)$ points, i.e. $K_Y^2 = 2(2g(S_1)-2)(2g(S_2)-2)$. Therefore for the quotient surface $X = S_1 \times S_2/G$ one has

$$K_X^2 = \frac{2(2g(S_1)-2)(2g(S_2)-2)}{|G|},$$

which gives the expression (16) for $\chi(\mathcal{O}_X)$.

Using these ingredients we can prove the following lemma.

LEMMA 2. *Let G be an arbitrary finite group and $X = S_1 \times S_2/G$ an unmixed Beauville surface isogenous to the product of two Riemann surfaces S_1 and S_2 of genera $(g(S_1), g(S_2)) = (p+1, q+1)$ for two prime numbers p and q. Then:*

(i) $p = q$;
(ii) $G = (\mathbb{Z}/n\mathbb{Z})^2$ *for some integer* n;
(iii) $S_1 \cong S_2 \cong F_n$, *the Fermat curve of degree* n.

PROOF. By formula (16) the fraction $\chi(\mathcal{O}_X) = pq/|G|$ is a natural number. The only possibility for G being non abelian is to be isomorphic to $\mathbb{Z}/q\mathbb{Z} \rtimes \mathbb{Z}/p\mathbb{Z}$, which can occur only if p divides $q-1$. We claim that in this case G does not admit a Beauville structure.

Indeed, since all $p-$groups (resp. $q-$groups) are conjugate, then any possible pair of generating triples (a_1, b_1, c_1) and (a_2, b_2, c_2) satisfying the compatibility condition (14) must have orders (p, p, p) and (q, q, q) respectively. Now the image $\overline{x} \in G/(\mathbb{Z}/q\mathbb{Z})$ of any element $x \in G$ of order q can only be the identity, and so $x \in \mathbb{Z}/q\mathbb{Z}$. In other words, no triple of elements of order q such as (a_2, b_2, c_2) can generate the whole group G. Therefore G must be abelian, and by [**7**] necessarily $p = q$ and $G = (\mathbb{Z}/p\mathbb{Z})^2$.

Now, arguing as in Example 2, we can deduce that both Riemann surfaces S_1 and S_2 are isomorphic to the Fermat curve of degree p. □

In fact there are no Fermat curves of genus $p+1$ for any prime $p > 5$. This is only because the genus of F_d is $g = (d-1)(d-2)/2$, which cannot equal $p+1$ for any prime $p > 5$.

THEOREM 1. *If $X = S_1 \times S_2/G$ is an unmixed Beauville surface with pair of genera $(g(S_1), g(S_2)) = (p+1, q+1)$, for prime numbers p and q, then $p = q = 5$ and X is isomorphic to the complex surface described in Example 1. In particular, this is the only Beauville surface reaching the minimum possible pair of genera $(6, 6)$.*

The next pair of genera (in the lexicographic order) for which there exists a Beauville surface is $(8, 49)$, therefore there are not Beauville surfaces with pair of genera $(6, g(S_2))$ or $(7, g(S_2))$ for any $g(S_2) > 6$.

PROOF. The first part of the theorem follows directly from the previous comments and the already mentioned fact that Beauville's original example described in Example 1 is the only Beauville surface with group $(\mathbb{Z}/5\mathbb{Z})^2$.

As for the second one we recall that the symmetric group $\mathfrak{S}_5$ is the only non-abelian group up to order 128 admitting a Beauville structure ([**2**]), the corresponding pair of genera being $(19, 21)$ (see [**14**]). Now, a list of all the groups G acting on Riemann surfaces of small genera so that the quotients are orbifolds of genus zero with three cone points is given in [**8**]. There are only five such groups of orders $|G| \geq 128$ acting on Riemann surfaces of genus 6 to 8. A computation carried out with MAGMA for these five groups shows that only the group $G = \mathrm{PSL}(2, 7)$ admits Beauville structures, among which the minimum pair of genera is $(8, 49)$

(two explicit pairs of hyperbolic triples of generators satisfying the compatibility condition (14) are given in [**14**]). □

4.2. Isomorphisms of unmixed Beauville surfaces. Let us suppose that there is an isomorphism f between two Beauville surfaces X and X'. By covering space theory we can lift f to an isomorphism between their universal coverings to obtain a commutative diagram as follows

$$\begin{array}{ccccccc} & & \mathbb{H}\times\mathbb{H} & \xrightarrow{\tilde{f}} & \mathbb{H}\times\mathbb{H} & & \\ & & \downarrow & & \downarrow & & \\ X & = & \dfrac{\mathbb{H}\times\mathbb{H}}{\Gamma_{12}} & \xrightarrow{f} & \dfrac{\mathbb{H}\times\mathbb{H}}{\Gamma'_{12}} & = & X' \end{array}$$

By a theorem of Cartan it is known that $\mathrm{Aut}(\mathbb{H}\times\mathbb{H}) = (\mathrm{Aut}(\mathbb{H})\times\mathrm{Aut}(\mathbb{H}))\rtimes\langle J\rangle$, where $\langle J\rangle$ is the group of order two generated by the automorphism $J(\mathrm{w}_1,\mathrm{w}_2) = (\mathrm{w}_2,\mathrm{w}_1)$ ([**17**],see also [**24**]). Therefore, there exist $\tilde{f}_1,\tilde{f}_2\in\mathrm{PSL}(2,\mathbb{R})$ such that

$$\tilde{f}(\mathrm{w}_1,\mathrm{w}_2) = \begin{cases} (\tilde{f}_1(\mathrm{w}_1),\tilde{f}_2(\mathrm{w}_2)), & \text{if } \tilde{f} \text{ does not interchange factors,} \\ (\tilde{f}_1(\mathrm{w}_2),\tilde{f}_2(\mathrm{w}_1)), & \text{if } \tilde{f} \text{ interchanges factors.} \end{cases} \tag{17}$$

Note that in the second case $\tilde{f}$ can be rewritten as $\tilde{f} = (\tilde{f}_1,\tilde{f}_2)\circ J$.

Proposition 4. *Let $\tilde{f}$ be as above. Then, perhaps after interchanging factors, one has:*

(i) $K'_i = \tilde{f}\,K_i\,\tilde{f}^{-1}$ *for* $i=1,2$, *and therefore* $K'_1\times K'_2 = \tilde{f}\,(K_1\times K_2)\,\tilde{f}^{-1}$;
(ii) $\Gamma'_i = \tilde{f}\,\Gamma_i\,\tilde{f}^{-1}$ *for* $i=1,2$;
(iii) $\tilde{f}$ *induces an isomorphism of twisted coverings between* $S_i\to S_i/G$ *and* $S'_i\to S'_i/G'$; *thus, in particular, an isomorphism between the groups* G *and* G'.

Proof. The proof of (i) is straightforward. Let us suppose first that $\tilde{f}$ does not interchange factors, so that we can write $\tilde{f} = (\tilde{f}_1,\tilde{f}_2)$ for some $\tilde{f}_1,\tilde{f}_2\in\mathrm{PSL}(2,\mathbb{R})$. If $\beta\in K_1$ then by (13) one has $(\beta,1)\in\Gamma_{12}$ and $\tilde{f}\,(\beta,1)\,\tilde{f}^{-1} = (\tilde{f}_1\beta\tilde{f}_1^{-1},1)\in\Gamma'_{12}$ so, again by (13), $\tilde{f}_1\beta\tilde{f}_1^{-1}\in K'_1$. Applying the same argument to the inverse $\tilde{f}^{-1}$ the result follows.

If $\tilde{f}$ does interchange factors, we can write it as $\tilde{f} = (\tilde{f}_1,\tilde{f}_2)\circ J$. Now, since $J^{-1} = J$, for any $\beta\in K_1$ we have

$$\begin{aligned} \tilde{f}\,(\beta,1)\,\tilde{f}^{-1}(\mathrm{w}_1,\mathrm{w}_2) &= (\tilde{f}_1,\tilde{f}_2)\circ J\circ(\beta,1)\circ J\circ(\tilde{f}_1^{-1},\tilde{f}_2^{-1})(\mathrm{w}_1,\mathrm{w}_2) = \\ &= (1,\tilde{f}_2\beta\tilde{f}_2^{-1})(\mathrm{w}_1,\mathrm{w}_2)\,, \end{aligned}$$

so $\tilde{f}_2\beta\tilde{f}_2^{-1}\in K'_2$ as before.

Finally, (ii) is obvious and (iii) follows directly from the previous points since $G = \Gamma_i/K_i$. □

We are now in position to understand when two pairs of defining triples give rise to isomorphic Beauville surfaces.

Proposition 5. *Two unmixed Beauville surfaces X and X' are isomorphic if and only if there exist $\delta_1,\delta_2\in\mathrm{PSL}(2,\mathbb{R})$, $\psi\in\mathrm{Aut}(G)$ and a permutation $\nu\in\mathfrak{S}_2$*

such that the following diagrams commute

$$\begin{array}{ccc} \Gamma_1 & \xrightarrow{\varphi_{\delta_1}} & \Gamma'_{\nu(1)} \\ {\scriptstyle\rho_1}\downarrow & & \downarrow{\scriptstyle\rho'_{\nu(1)}} \\ G & \xrightarrow{\psi} & G \end{array} \qquad \begin{array}{ccc} \Gamma_2 & \xrightarrow{\varphi_{\delta_2}} & \Gamma'_{\nu(2)} \\ {\scriptstyle\rho_2}\downarrow & & \downarrow{\scriptstyle\rho'_{\nu(2)}} \\ G & \xrightarrow{\psi} & G \end{array} \tag{18}$$

i.e. such that $\psi \circ \rho_i = \rho'_{\nu(i)} \circ \varphi_{\delta_i}$.

Proof. If f does not interchange factors, then we can write $\tilde{f} = (\tilde{f}_1, \tilde{f}_2)$ and take $\tilde{f}_1, \tilde{f}_2$ as δ_1, δ_2. Now, the proof of Proposition 4 shows that the first (resp. the second) diagram commutes if we take as ψ the group automorphism ψ_1 (resp. ψ_2) that sends $\rho_1(\gamma_1)$ to $\rho'_{\nu(1)}(\delta_1\gamma_1\delta_1^{-1})$ (resp. $\rho_2(\gamma_2)$ to $\rho'_{\nu(2)}(\delta_2\gamma_2\delta_2^{-1})$). But $\psi_1 = \psi_2$ because, as $\tilde{f}$ conjugates Γ_{12} into Γ'_{12}, the equality $\rho_1(\gamma_1) = \rho_2(\gamma_2)$ implies that $\rho'_{\nu(1)}(\delta_1\gamma_1\delta_1^{-1}) = \rho'_{\nu(2)}(\delta_2\gamma_2\delta_2^{-1})$.

Conversely, if the conditions hold, the uniformizing groups of X and X' are readily seen to be conjugate, for we have

$$\begin{aligned} \Gamma'_{12} &= \{(\gamma'_1, \gamma'_2) \in \Gamma'_1 \times \Gamma'_2 : \rho'_1 \circ \varphi_{\delta_1}(\varphi_{\delta_1}^{-1}(\gamma'_1)) = \rho'_2 \circ \varphi_{\delta_2}(\varphi_{\delta_2}^{-1}(\gamma'_2))\} = \\ &= \{(\gamma'_1, \gamma'_2) \in \Gamma'_1 \times \Gamma'_2 : \rho_1(\varphi_{\delta_1}^{-1}(\gamma'_1)) = \rho_2(\varphi_{\delta_2}^{-1}(\gamma'_2))\} = \\ &= \{(\varphi_{\delta_1}(\gamma_1), \varphi_{\delta_2}(\gamma_2)) \in \Gamma'_1 \times \Gamma'_2 : \rho_1(\gamma_1) = \rho_2(\gamma_2)\} = \\ &= (\varphi_{\delta_1} \times \varphi_{\delta_2})(\Gamma_{12}) = (\delta_1, \delta_2) \circ \Gamma_{12} \circ (\delta_1, \delta_2)^{-1}. \end{aligned}$$

In the case when f is factor-reversing we have $\rho'_2 \circ \varphi_{\delta_1} = \psi \circ \rho_1$ and $\rho'_1 \circ \varphi_{\delta_2} = \psi \circ \rho_2$, and the proof goes word for word as above. ☐

We can translate this proposition into conditions on the pairs of triples of generators of G for their corresponding Beauville surfaces to be isomorphic.

Corollary 2. *Let* $q = (a_1, b_1, c_1; a_2, b_2, c_2)$ *and* $q' = (a'_1, b'_1, c'_1; a'_2, b'_2, c'_2)$ *be two Beauville structures on* G. *Then the Beauville surfaces corresponding to* q *and* q' *are isomorphic if and only if there exists* $\psi \in \mathrm{Aut}(G)$ *and* $\nu \in \mathfrak{S}_2$ *such that*

$$\psi(a_i, b_i, c_i) \equiv (a'_{\nu(i)}, b'_{\nu(i)}, c'_{\nu(i)}) \bmod \mathrm{I}(G; l'_{\nu(i)}, m'_{\nu(i)}, n'_{\nu(i)}), \quad i = 1, 2. \tag{19}$$

Moreover, the corresponding uniformizing groups are conjugate by means of any element $(\delta_1, \delta_2) \in \mathrm{Aut}(\mathbb{H}) \times \mathrm{Aut}(\mathbb{H})$ *fitting into* (18).

Proof. First note that the type of the triples is preserved by isomorphisms, so $(l_i, m_i, n_i) = (l'_{\nu(i)}, m'_{\nu(i)}, n'_{\nu(i)})$.

Now, by Remark 2, the condition (19) is equivalent to the existence of elements $\delta_1, \delta_2 \in \mathrm{PSL}(2, \mathbb{R})$ yielding by conjugation isomorphisms $\varphi_{\delta_i} : \Gamma_i \longrightarrow \Gamma'_{\nu(i)}$, such that $\rho'_{\nu(i)} \circ \varphi_{\delta_i} = \psi \circ \rho_i$. The corollary is then a direct consequence of Proposition 5. ☐

By the comments above we have the following

Corollary 3. *The following are invariants of the isomorphism class of an unmixed Beauville surface* $X = S_1 \times S_2/G$:

(i) *the group* G;
(ii) *the bitype* $((l_1, m_1, n_1), (l_2, m_2, n_2))$;
(iii) *the twisted isomorphism class of the orbifolds* S_i/G, *hence the Riemann surfaces* S_i *themselves.*

4.3. Automorphisms of unmixed Beauville surfaces. In this section we will study the group of automorphisms of unmixed Beauville surfaces. If we denote by $\Gamma_{12} < \mathrm{Aut}(\mathbb{H}) \times \mathrm{Aut}(\mathbb{H})$ the group uniformizing such a Beauville surface X, as described in (12) and (13), then of course $\mathrm{Aut}(X) \cong N(\Gamma_{12})/\Gamma_{12}$, where $N(\Gamma_{12})$ stands for the normalizer of Γ_{12} in $\mathrm{Aut}(\mathbb{H} \times \mathbb{H})$.

Consider first the subgroup $N(\Gamma_{12}) \cap (\Gamma_1 \times \Gamma_2)$. We have the following result.

Lemma 3. *The rule*

$$\begin{array}{rccc} \phi : & N(\Gamma_{12}) \cap (\Gamma_1 \times \Gamma_2) & \longrightarrow & Z(G) \\ & (\gamma_1, \gamma_2) & \longmapsto & \rho_2(\gamma_2)^{-1}\rho_1(\gamma_1) \end{array}$$

defines an epimorphism whose kernel is Γ_{12}. *Here, as usual,* $Z(G)$ *stands for the centre of* G.

Proof. We first observe that an element $(\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2$ normalizes Γ_{12} if and only if for every $g \in G$ one has

$$\rho_1(\gamma_1) g \rho_1(\gamma_1)^{-1} = \rho_2(\gamma_2) g \rho_2(\gamma_2)^{-1}, \tag{20}$$

i.e. $\rho_2(\gamma_2)^{-1}\rho_1(\gamma_1) \in Z(G)$. This shows that the map ϕ is well defined.

Now it is easy to see that ϕ is a homomorphism. Indeed

$$\begin{aligned} \phi\left((\gamma_1, \gamma_2) \circ (\gamma_1', \gamma_2')\right) &= \phi\left(\gamma_1\gamma_1', \gamma_2\gamma_2'\right) = \rho_2(\gamma_2')^{-1}\rho_2(\gamma_2)^{-1}\rho_1(\gamma_1)\rho_1(\gamma_1') = \\ &= \rho_2(\gamma_2')^{-1}\phi\left(\gamma_1, \gamma_2\right)\rho_1(\gamma_1') = \phi\left(\gamma_1, \gamma_2\right) \cdot \phi\left(\gamma_1', \gamma_2'\right). \end{aligned}$$

On the other hand, if $\rho_1(\beta) = h \in Z(G)$ then the element $(\beta, 1)$ clearly satisfies the relation (20) and therefore it is a preimage of h.

Finally, we see that $\phi(\gamma_1, \gamma_2) = 1$ if and only if $\rho_1(\gamma_1) = \rho_2(\gamma_2)$, that is if and only if $(\gamma_1, \gamma_2) \in \Gamma_{12}$. □

Now we can prove the following

Theorem 2. *Let* X *be an unmixed Beauville surface with Beauville group* G. *The group* $Z(G)$ *is naturally identified with a subgroup of* $\mathrm{Aut}(X)$ *of index dividing 72. More precisely, let* X *have bitype* $((l_1, m_1, n_1), (l_2, m_2, n_2))$, *and consider natural numbers* ε, κ_1, *and* κ_2 *where* ε *equals 2 if the types* (l_1, m_1, n_1) *and* (l_2, m_2, n_2) *agree and 1 otherwise, and* κ_i *equals 6, 2 or 1 depending on whether the type* (l_i, m_i, n_i) *has three, two or no repeated orders. Then there exists a natural number* N *dividing* $\varepsilon \cdot \kappa_1 \cdot \kappa_2$ *such that*

$$|\mathrm{Aut}(X)| = N \cdot |Z(G)|.$$

In particular, if $\kappa_1 = \kappa_2 = \varepsilon = 1$ *we have that* $\mathrm{Aut}(X) \cong Z(G)$.

Proof. The previous lemma permits us to regard $Z(G)$ as a subgroup of $\mathrm{Aut}(X)$ via the identification

$$Z(G) \cong \frac{N(\Gamma_{12}) \cap (\Gamma_1 \times \Gamma_2)}{\Gamma_{12}} \leq \mathrm{Aut}(X).$$

Consider the intersections

$$\begin{aligned} N_0(\Gamma_{12}) &= N(\Gamma_{12}) \cap (\mathrm{Aut}(\mathbb{H}) \times \mathrm{Aut}(\mathbb{H})) \qquad \text{and} \\ N_1(\Gamma_{12}) &= N_0(\Gamma_{12}) \cap (\Gamma_1 \times \Gamma_2) = N(\Gamma_{12}) \cap (\Gamma_1 \times \Gamma_2). \end{aligned}$$

Using the identity $|\mathrm{Aut}(X)| = [N(\Gamma_{12}) : \Gamma_{12}]$ one gets the following equality

$$|\mathrm{Aut}(X)| = [N(\Gamma_{12}) : N_0(\Gamma_{12})] \cdot [N_0(\Gamma_{12}) : N_1(\Gamma_{12})] \cdot [N_1(\Gamma_{12}) : \Gamma_{12}].$$

Now, $\varepsilon := [N(\Gamma_{12}) : N_0(\Gamma_{12})] \leq 2$ and $[N_1(\Gamma_{12}) : \Gamma_{12}] = |Z(G)|$.

On the other hand, clearly one has $N_0(\Gamma_{12}) < N(\Gamma_1) \times N(\Gamma_2)$, and therefore $[N_0(\Gamma_{12}) : N_1(\Gamma_{12})]$ divides $[N(\Gamma_1) \times N(\Gamma_2) : \Gamma_1 \times \Gamma_2]$. If we write $\kappa_i := |N(\Gamma_i)/\Gamma_i|$, then $[N(\Gamma_1) \times N(\Gamma_2) : \Gamma_1 \times \Gamma_2] = \kappa_1 \cdot \kappa_2$ and the result follows from (8). □

The above bounds are actually sharp, as shown by examples by Y. Fuertes ([**12**]) and by G. A. Jones in [**22**]. This last paper contains most of what is known about the automorphism groups of unmixed Beauville surfaces.

Example 3. For Beauville's original surface with group $G = (\mathbb{Z}/5\mathbb{Z})^2$ and bitype $((5,5,5),(5,5,5))$, the automorphism group is a semidirect product of $Z(G) = G$ by $\mathbb{Z}/3\mathbb{Z}$ ([**19**]), and therefore $|\mathrm{Aut}(X)| = 3 \cdot |Z(G)| = 75$.

Remark 4. An interesting family of Beauville surfaces with trivial automorphism group can be obtained as follows. Everitt has shown in [**11**] that for every hyperbolic signature (l,m,n) there are triangle $G-$coverings of type (l,m,n) with $G = \mathfrak{A}_r$, the alternating group on r elements, for almost every r. As a consequence for any pair of hyperbolic signatures $(l,m,n),(p,q,r)$ such that the integers lmn and pqr are coprime (so that the compatibility condition holds), we can construct Beauville surfaces of this bitype with Beauville group $G = \mathfrak{A}_r$. Since for $r \geq 4$ the centre of $\mathfrak{A}_r$ is trivial, if the orders of each of the two signatures are all different we have $\mathrm{Aut}(X) = \{\mathrm{Id}\}$.

5. Uniformization of Beauville surfaces: mixed case

We focus our attention now on the mixed case. Recall that a mixed Beauville surface is a surface of the form $X = S_1 \times S_2/G$, where G is a finite group acting freely on $S_1 \times S_2$ so that the index two subgroup $G^0 \lhd G$ of factor-preserving elements of G acts on each of the two Riemann surfaces in such a way that the projections $S_i \longrightarrow S_i/G^0 \cong \widehat{\mathbb{C}}$ ramify over three values. Note that if $g \in G \backslash G^0$ then $G = \langle G^0, g \rangle$ and, moreover, the action of g defines a factor-reversing automorphism of the associated unmixed Beauville surface $X^0 = S_1 \times S_2/G^0$. By Proposition 4, such an element g induces an isomorphism between the orbifolds S_1/G^0 and S_2/G^0. It follows that in this case $S_1 \cong S_2$, and that the corresponding triangle groups Γ_1 and Γ_2 are both equal to the group $\Gamma = \Gamma(l,m,n)$. As a consequence in the mixed case instead of the bitype we will simply call (l,m,n) the *type of* X.

Uniformization theory tells us that there is a group $\Gamma_{12} < \mathrm{Aut}(\mathbb{H} \times \mathbb{H})$ such that $X = \mathbb{H} \times \mathbb{H}/\Gamma_{12}$ and $X^0 = \mathbb{H} \times \mathbb{H}/\Gamma_{12}^0$ where $\Gamma_{12}^0 < \Gamma \times \Gamma$ is the index two subgroup of Γ_{12} consisting of the factor-preserving elements. Therefore we have exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K_1 \times K_2 & \longrightarrow & \Gamma_{12} & \stackrel{\rho}{\longrightarrow} & G & \longrightarrow & 1 \\ 1 & \longrightarrow & K_1 \times K_2 & \longrightarrow & \Gamma_{12}^0 & \stackrel{\rho^0}{\longrightarrow} & G^0 & \longrightarrow & 1 \end{array}$$

where $\rho^0((\gamma_1,\gamma_2)) = \rho_1(\gamma_1) = \rho_2(\gamma_2)$ and $\rho|_{\Gamma_{12}^0} = \rho^0$. In particular the epimorphism ρ is determined by ρ^0 together with the image $\rho(\mathfrak{h}) = h \in G$ of any chosen element $\mathfrak{h} \in \Gamma_{12} \setminus \Gamma_{12}^0$.

Note that each element $\mathfrak{h} \in \Gamma_{12} \setminus \Gamma_{12}^0$ can be written as $\mathfrak{h} = (\beta_1,\beta_2) \circ J$ where $\beta_1, \beta_2 \in \mathrm{Aut}(\mathbb{H})$. Now, as $\mathfrak{h}$ must normalize Γ_{12}^0, for every element $(\gamma_1,\gamma_2) \in \Gamma_{12}^0$

we have

$$\mathfrak{h}\circ(\gamma_1,\gamma_2)\circ\mathfrak{h}^{-1} = (\beta_1,\beta_2)\circ J\circ(\gamma_1,\gamma_2)\circ J\circ(\beta_1^{-1},\beta_2^{-1}) = (\beta_1\gamma_2\beta_1^{-1},\beta_2\gamma_1\beta_2^{-1})\in\Gamma_{12}^0\,. \tag{21}$$

It follows that $\beta_1,\beta_2\in N(\Gamma)$, the normalizer of $\Gamma=\Gamma(l,m,n)$.

We can use these facts to get a criterion for mixed surfaces analogous to the one established in Corollary 1 for the unmixed ones. As in the unmixed case we will say that a finite group G admits a *mixed Beauville structure* if there exists an action of G on the product of two Riemann surfaces defining a mixed Beauville surface.

Proposition 6. *A finite group G admits a mixed Beauville structure if and only if there exist an index two subgroup $G^0\lhd G$ and elements $a,b,c\in G^0$ such that the following conditions hold:*

(i) *(a,b,c) is a hyperbolic triple of generators of G^0;*
(ii) *$h^2\neq\mathrm{Id}$, for every $h\in G\setminus G^0$;*
(iii) *there exists $g\in G\setminus G^0$ such that $\Sigma(a,b,c)\cap\Sigma(gag^{-1},gbg^{-1},gcg^{-1})=\{\mathrm{Id}\}$.*

Proof. Suppose that the group G admits a mixed Beauville structure and write $X=S_1\times S_2/G$ for the corresponding mixed Beauville surface. The existence of a triple as in condition (i) follows from the fact that, if G^0 is the subgroup of factor-preserving elements, then $X^0=S_1\times S_2/G^0$ is an unmixed Beauville surface. Actually, by the previous sections, we have two obvious such triples at our disposal, namely $(a_1,b_1,c_1)=(\rho_1(x),\rho_1(y),\rho_1(z))$ and $(a_2,b_2,c_2)=(\rho_2(x),\rho_2(y),\rho_2(z))$ where, again, $\rho_i:\Gamma\longrightarrow G$ is the epimorphism associated to the orbifold S_i/G^0, $i=1,2$. We claim that $\Sigma(a_2,b_2,c_2)=\Sigma(ga_1g^{-1},gb_1g^{-1},gc_1g^{-1})$ for some element $g\in G\setminus G^0$. This would clearly imply condition (iii).

To prove this, let $\mathfrak{h}=(\beta_1,\beta_2)\circ J$ be an element in $\Gamma_{12}\setminus\Gamma_{12}^0$ where, as before, Γ_{12} and Γ_{12}^0 are the uniformizing groups of X and X^0, and put $\rho(\mathfrak{h})=h\in G\setminus G^0$. Now choose elements $x',y',z'\in\Gamma$ such that $a_1=\rho_1(x)=\rho_2(x')$, $b_1=\rho_1(y)=\rho_2(y')$ and $c_1=\rho_1(z)=\rho_2(z')$, so that $(x,x'),(y,y'),(z,z')\in\Gamma_{12}^0$. Then, formula (21) applied to $(\gamma_1,\gamma_2)=(x,x')$, (y,y') and (z,z') gives

$$\begin{aligned}
ha_1h^{-1} &= \rho(\beta_1x'\beta_1^{-1},\beta_2x\beta_2^{-1})=\rho_1(\beta_1x'\beta_1^{-1})=\rho_2(\beta_2x\beta_2^{-1})\,,\\
hb_1h^{-1} &= \rho(\beta_1y'\beta_1^{-1},\beta_2y\beta_2^{-1})=\rho_1(\beta_1y'\beta_1^{-1})=\rho_2(\beta_2y\beta_2^{-1})\,,\\
hc_1h^{-1} &= \rho(\beta_1z'\beta_1^{-1},\beta_2z\beta_2^{-1})=\rho_1(\beta_1z'\beta_1^{-1})=\rho_2(\beta_2z\beta_2^{-1})\,.
\end{aligned}$$

As $\beta_2\in N(\Gamma)$, by (9) we can write $\beta_2=\eta\lambda_i$, for some $\eta\in\Gamma$ and λ_i, $i=0,\ldots,5$, as in Table 1. Therefore we have $\beta_2\gamma\beta_2^{-1}=\eta\lambda_i\gamma\lambda_i^{-1}\eta^{-1}$ for any $\gamma\in\Gamma$. In particular, if we denote $\rho_2(\eta)=k^{-1}\in G^0$, the three relations above give the following three identities

$$\begin{aligned}
kha_1h^{-1}k^{-1} &= \rho_2(\lambda_ix\lambda_i^{-1})=\rho_2(\tilde\sigma_i(x))\,,\\
khb_1h^{-1}k^{-1} &= \rho_2(\lambda_iy\lambda_i^{-1})=\rho_2(\tilde\sigma_i(y))\,,\\
khc_1h^{-1}k^{-1} &= \rho_2(\lambda_iz\lambda_i^{-1})=\rho_2(\tilde\sigma_i(z))\,.
\end{aligned}$$

Now setting $g=kh$ we find that the epimorphisms ρ_2 and ρ_1 are related by the formula $\rho_2\circ\tilde\sigma_i(\gamma)=g\rho_1(\gamma)g^{-1}$. In particular, by formula (11) we have that

$$\begin{aligned}
\Sigma(gag^{-1},gbg^{-1},gcg^{-1}) &= \Sigma(\rho_2(\lambda_ix\lambda_i^{-1}),\rho_2(\lambda_iy\lambda_i^{-1}),\rho_2(\lambda_iz\lambda_i^{-1}))=\\
&= \Sigma(\sigma_i(\rho_2(x),\rho_2(y),\rho_2(z)))=\Sigma(\sigma_i(a_2,b_2,c_2))=\\
&= \Sigma(a_2,b_2,c_2)\,,
\end{aligned}$$

where the last equality follows from the fact that by definition (see (10)) the transformations σ_i preserve the union of the conjugacy classes of the three elements a_2, b_2, c_2.
To check condition (ii) first observe that an element $h \in G \setminus G^0$ fixes some point on the product $S_1 \times S_2$ if and only if its square $h^2 \in G^0$ does. This is because if h is defined by $h(P_1, P_2) = (h_1(P_2), h_2(P_1))$, and its square

$$h^2(P_1, P_2) = (h_1 h_2(P_1), h_2 h_1(P_2))$$

fixes a point (P_1, P_2), then h fixes the point $(P_1, h_2(P_1))$. Now condition (ii) is a consequence of the fact that the action of G is free.

For the converse we start by noting that conditions (i) and (iii) ensure the existence of an unmixed structure in G^0, given by the pairs of triples (a, b, c) and $(gag^{-1}, gbg^{-1}, gcg^{-1})$, and therefore of the corresponding unmixed Beauville surface $X^0 = S_1 \times S_2/G^0 \cong \mathbb{H} \times \mathbb{H}/\Gamma_{12}^0$. What remains to be done is to extend this action to $G \setminus G^0$ in a way that there are no fixed points, or equivalently to extend the action of Γ_{12}^0 on $\mathbb{H} \times \mathbb{H}$ to a suitable group Γ_{12}. In this case, the special relationship between the two defining triples implies the following relation between their associated epimorphisms ρ_1 and ρ_2:

$$\rho_1(\gamma_2) = g^{-1}\rho_2(\gamma_2)g, \quad \text{for any } \gamma_2 \in \Gamma. \tag{22}$$

In particular, if $\rho_1(\tau) = g^2$, then $\rho_2(\tau) = g^2$ too and therefore $(\tau, \tau) \in \Gamma_{12}^0$. Let us define $\Gamma_{12} := \langle \Gamma_{12}^0, \mathfrak{g} \rangle$ with $\mathfrak{g} = (\tau, 1) \circ J$. We claim that Γ_{12}^0 is a subgroup of index two, hence normal, of Γ_{12}. To see this it is enough to check that $\mathfrak{g}^2 \in \Gamma_{12}^0$ and that $\mathfrak{g}$ normalizes Γ_{12}^0. The first property is obvious, in fact $\mathfrak{g}^2 = (\tau, \tau) \in \Gamma_{12}^0$. As for the second one, we have to see that for every $(\gamma_1, \gamma_2) \in \Gamma_{12}^0$ the element

$$\mathfrak{g} \circ (\gamma_1, \gamma_2) \circ \mathfrak{g}^{-1} = (\tau, 1) \circ J \circ (\gamma_1, \gamma_2) \circ J \circ (\tau^{-1}, 1) = (\tau\gamma_2\tau^{-1}, \gamma_1) \tag{23}$$

lies in Γ_{12}^0. But, by (22), one has

$$\rho_1(\tau\gamma_2\tau^{-1}) = g^2\rho_1(\gamma_2)g^{-2} = g\rho_2(\gamma_2)g^{-1} = g\rho_1(\gamma_1)g^{-1} = \rho_2(\gamma_1)$$

as required.

Now let $\rho : \Gamma_{12} \longrightarrow G$ be the epimorphism determined by

$$\begin{array}{rccl} \rho: & \Gamma_{12} & \longrightarrow & G \\ & (\gamma_1, \gamma_2) & \longmapsto & \rho^0(\gamma_1, \gamma_2), \quad \text{if } (\gamma_1, \gamma_2) \in \Gamma_{12}^0 \\ & \mathfrak{g} & \longmapsto & g \end{array}$$

It is easy to check that ρ defines a homomorphism. In fact, using (23), one finds that $\mathfrak{g} \circ (\gamma_1', \gamma_2') \circ \mathfrak{g}^{-1} = (\tau\gamma_2'\tau^{-1}, \gamma_1')$ and now, using (22), we can write

$$\begin{aligned} \rho((\gamma_1, \gamma_2) \circ \mathfrak{g}) \cdot \rho((\gamma_1', \gamma_2') \circ \mathfrak{g}) &= \rho_1(\gamma_1)g\rho_1(\gamma_1')g = \rho_1(\gamma_1)\rho_2(\gamma_1')g^2 = \\ &= \rho((\gamma_1, \gamma_2) \circ (\tau\gamma_2'\tau^{-1}, \gamma_1') \circ \mathfrak{g}^2) = \\ &= \rho((\gamma_1, \gamma_2) \circ \mathfrak{g} \circ (\gamma_1', \gamma_2') \circ \mathfrak{g}) \end{aligned}$$

as desired.

Clearly, the kernel of ρ is the same as the kernel of ρ^0, namely a product of Fuchsian groups $K_1 \times K_2$ uniformizing the product of Riemann surfaces $S_1 \times S_2$. Therefore the mixed Beauville surface we are looking for is

$$X = \frac{\mathbb{H} \times \mathbb{H}}{\Gamma_{12}} = \frac{S_1 \times S_2}{G},$$

which obviously has

$$X^0 = \frac{\mathbb{H} \times \mathbb{H}}{\Gamma_{12}^0} = \frac{S_1 \times S_2}{G^0}$$

as underlying unmixed Beauville surface.

To complete the proof it only remains to observe that also the elements $h \in G \setminus G^0$ (i.e. the factor-reversing ones) have to act freely on $S_1 \times S_2$ since, as noted earlier, otherwise $h^2 \in G^0$ would also fix some point, which is a contradiction. □

Remark 5. It is important to observe that if, in the construction above, instead of the element g we use another element $g' \in G \setminus G^0$ satisfying condition (iii) in Proposition 6, then the mixed Beauville surface X' so obtained will be isomorphic to X.

In fact, write $g' = h_0 g \in G \setminus G^0$ for some $h_0 \in G^0$. We claim that the uniformizing groups $\Gamma_{12} = \langle \Gamma_{12}^0, (\tau, 1) \circ J \rangle$ and $\Gamma'_{12} = \langle \Gamma'^0_{12}, (\tau', 1) \circ J \rangle$, defined by the corresponding epimorphisms ρ_i and ρ'_i, are conjugate by means of an element of the form $(1, \eta)$, where η is any element of Γ such that $\rho_2(\eta) = h_0^{-1}$. To see this, first note that $(1, \eta)$ conjugates Γ_{12}^0 into Γ'^0_{12}. In fact, since $\rho'_2(\gamma) = h_0 \rho_2(\gamma) h_0^{-1}$, for any $(\gamma_1, \gamma_2) \in \Gamma_{12}^0$ one has

$$(1, \eta)(\gamma_1, \gamma_2)(1, \eta)^{-1} = (\gamma_1, \eta\gamma_2\eta^{-1}) \in \Gamma'^0_{12} ,$$

because $\rho'_2(\eta\gamma_2\eta^{-1}) = h_0 \rho_2(\eta\gamma_2\eta^{-1}) h_0^{-1} = \rho_2(\gamma_2) = \rho_1(\gamma_1)$.

Now, since $[\Gamma_{12} : \Gamma_{12}^0] = [\Gamma'_{12} : \Gamma'^0_{12}] = 2$, to prove our claim it is enough to find an element $\mathfrak{p} \in \Gamma_{12} \setminus \Gamma_{12}^0$ whose conjugate by $(1, \eta)$ lies in $\Gamma'_{12} \setminus \Gamma'^0_{12}$. For instance, take any element of the form $(\zeta, \eta^{-1}) \in \Gamma_{12}^0$ and let $\mathfrak{p} = (\zeta, \eta^{-1}) \circ (\tau, 1) \circ J \in \Gamma_{12} \setminus \Gamma_{12}^0$. Then

$$(1, \eta) \circ \mathfrak{p} \circ (1, \eta)^{-1} = (\zeta\tau\eta^{-1}, 1) \circ J = (\tau', 1) \circ J ,$$

the last identity because

$$\rho_1(\zeta\tau\eta^{-1}) = \rho_1(\zeta)\rho_1(\tau)\rho_1(\eta)^{-1} = h_0 g^2 g^{-1} \rho_2(\eta)^{-1} g = h_0 g^2 g^{-1} h_0 g = (h_0 g)^2 = g'^2.$$

Due to the remark above we can refer to a mixed Beauville structure on G simply by giving a quadruple $(G^0; a, b, c)$ satisfying the conditions in Proposition 6, without need to mention any particular element $g \in G \setminus G^0$.

5.1. Some restrictions to the existence of mixed Beauville surfaces. There are some obvious conditions that groups admitting mixed Beauville structures must satisfy. For instance, simple groups cannot do so, as they do not possess index two subgroups. Likewise, the symmetric groups $\mathfrak{S}_n$ do not admit mixed Beauville structures either. This is because the only subgroup of $\mathfrak{S}_n$ of index two is the alternating group $\mathfrak{A}_n$, and $\mathfrak{S}_n \setminus \mathfrak{A}_n$ contains plenty of elements of order two, a fact which violates condition (ii) in Proposition 6. Another family of groups which cannot admit mixed Beauville structures is the abelian ones (see [**2**], Theorem 4.3).

The next result included in [**13**] exhibits another restriction of this sort.

Proposition 7. *Let G be a group admitting a Beauville structure. Then the order of any element of $G \setminus G^0$ is divisible by 4. In particular, the order $|G|$ of G is a multiple of 4.*

Proof. Let $g \in G \setminus G^0$ an element of order k. If k is an odd natural number then g^k is still factor-reversing, thus different from the identity. Therefore k is necessarily even. Now if $k = 2d$, then $(g^d)^2 = 1$ which by condition (ii) in Proposition 6 implies that $g^d \in G^0$, which in turn implies that d is even. □

Next we give a restriction on the genus of the Riemann surfaces that can arise in the construction of mixed Beauville surfaces.

Since both Riemann surfaces S_1, S_2 intervening in the construction of a mixed Beauville surface are isomorphic to the same Riemann surface $S \cong S_1 \cong S_2$, using the formulae (15) and (16) for the Euler–Poincaré characteristic and the holomorphic Euler characteristic we get

$$\chi(\mathcal{O}_X) = \frac{\chi(X)}{4} = \frac{(g(S)-1)^2}{|G|} = \frac{(g(S)-1)^2}{2|G^0|} \in \mathbb{N},$$

where $g(S)$ is the genus of the Riemann surface S. Thus, in particular, $g(S)$ is odd. This formula already tells us that $(g(S)-1)^2 \geq |G|$.

On the other hand, by the Riemann–Hurwitz formula we have

$$2g(S) - 2 = |G^0| \left(1 - \left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n}\right)\right),$$

where (l, m, n) is the signature of the $G-$covering S. Furthermore, it is known that $1/42 \leq 1 - (1/l + 1/m + 1/n) < 1$ and therefore, from the last two formulae we can deduce that

$$\max\left\{\sqrt{|G|} + 1, \frac{|G|}{168} + 1\right\} \leq g(S) < \frac{|G|}{4} + 1\,. \tag{24}$$

Now it is known that no group of order smaller than 256 admits a mixed Beauville structure. In fact, in [**4**] it is proved that there are two groups of order 256 admitting a mixed Beauville structure of type $(4, 4, 4)$, whose corresponding Riemann surfaces have genus 17. This fact together with the lower bound in (24) leads to the following.

Corollary 4. *Let $X = S \times S/G$ be a mixed Beauville surface. Then $g(S)$ is an odd number ≥ 17 and this bound is sharp.*

Proof. We already noted that $g(S)$ has to be odd. Moreover, the comments above together with the relation (24) imply that $g(S) \geq \max\left\{\sqrt{256} + 1, \frac{256}{168} + 1\right\} = 17$. □

5.2. Isomorphisms of mixed Beauville surfaces. Let us consider two mixed Beauville surfaces $X = S \times S/G$ and $X' = S' \times S'/G'$, associated to mixed Beauville structures $(G^0; a, b, c)$ and $(G'^0; a', b', c')$, and having underlying unmixed Beauville surfaces X^0 and X'^0, respectively.

Suppose $f : X \longrightarrow X'$ is an isomorphism. Let $\tilde{f} : \mathbb{H} \times \mathbb{H} \longrightarrow \mathbb{H} \times \mathbb{H}$ be its lift to the universal cover and $f_* : \Gamma_{12} \longrightarrow \Gamma'_{12}$ the group isomorphism induced by $\tilde{f}$. Clearly the restriction $f_*|_{\Gamma^0_{12}}$ gives an isomorphism between Γ^0_{12} and Γ'^0_{12}. In particular f lifts to an isomorphism $f^0 : X^0 \longrightarrow X'^0$ and we have the following commutative diagram

$$\begin{array}{ccccccc} & & \mathbb{H}\times\mathbb{H} & \xrightarrow{\tilde f} & \mathbb{H}\times\mathbb{H} & & \\ & & \downarrow & & \downarrow & & \\ X^0 & = & \dfrac{\mathbb{H}\times\mathbb{H}}{\Gamma_{12}^0} & \xrightarrow{f^0} & \dfrac{\mathbb{H}\times\mathbb{H}}{\Gamma_{12}'^0} & = & X'^0 \\ & & \downarrow & & \downarrow & & \\ X & = & \dfrac{\mathbb{H}\times\mathbb{H}}{\Gamma_{12}} & \xrightarrow{f} & \dfrac{\mathbb{H}\times\mathbb{H}}{\Gamma_{12}'} & = & X' \end{array}$$

Moreover, by Proposition 4, $\tilde f$ sends $K_1\times K_2$ to $K_1'\times K_2'$, therefore it induces an isomorphism ψ between $G\cong\Gamma_{12}/K_1\times K_2$ and $G'\cong\Gamma_{12}'/K_1'\times K_2'$ which restricts to an isomorphism between $G^0\cong\Gamma_{12}^0/K_1\times K_2$ and $G'^0\cong\Gamma_{12}'^0/K_1'\times K_2'$.

By pre-composition with an element of Γ_{12} if necessary, we can always assume that $\tilde f$ is factor-preserving. Then, with the same notation as in section 4.2 (except that here G^0 plays the role of the group we denoted there by G), one has (see (19))

$$\psi(a,b,c)\equiv(a',b',c') \bmod \mathrm{I}(G'^0;l,m,n)\,. \tag{25}$$

Conversely, suppose that there exists an isomorphism $\psi: G\longrightarrow G'$, with $\psi(G^0)=G'^0$, such that $\psi(a,b,c)\equiv(a',b',c') \bmod \mathrm{I}(G'^0;l,m,n)$. We claim that under these circumstances the groups Γ_{12} and Γ_{12}' uniformizing the mixed Beauville surfaces corresponding to the quadruples $(G^0;a,b,c)$ and $(G'^0;a',b',c')$ are conjugate. We start by noting that their associated index two subgroups Γ_{12}^0 and $\Gamma_{12}'^0$ are conjugate. This follows from the fact that the pairs of triples defining the epimorphisms ρ_1, ρ_2 and ρ_1', ρ_2' which determine the groups Γ_{12}^0 and $\Gamma_{12}'^0$ are precisely $(a,b,c;gag^{-1},gbg^{-1},gcg^{-1})$ and $(a',b',c';g'a'g'^{-1},g'b'g'^{-1},g'c'g'^{-1})$ for arbitrary elements $g\in G\setminus G^0$ and $g'\in G'\setminus G'^0$ (see Proposition 6); and it is easy to see that the relation (25) between the triples (a,b,c) and (a',b',c') implies a similar relation between the triples $(gag^{-1},gbg^{-1},gcg^{-1})$ and $(g'a'g'^{-1},g'b'g'^{-1},g'c'g'^{-1})$. More precisely, if we have an identity of the form $\rho_1'\circ\varphi_\delta=\psi\circ\rho_1$ for some $\delta\in\mathrm{PSL}(2,\mathbb{R})$, which by Remark 2 is what (25) means, and we put $g'=\psi(g)\in G\setminus G^0$ then

$$\rho_2'\circ\varphi_\delta=\psi\circ\rho_2$$

since, for any $\gamma\in\Gamma$, one has

$$\rho_2'(\varphi_\delta(\gamma))=g'\rho_1'(\varphi_\delta(\gamma))g'^{-1}=g'\psi(\rho_1(\gamma))g'^{-1}=\psi(g\rho_1(\gamma)g^{-1})=\psi(\rho_2(\gamma))\,.$$

By Corollary 2, this implies that the subgroups Γ_{12}^0 and $\Gamma_{12}'^0$ are conjugate by means of the element (δ,δ).

Now consider an element $\mathfrak{g}=(\tau,1)\circ J\in\Gamma_{12}$ such that $\rho_1(\tau)=\rho_2(\tau)=g^2$, and an element $\mathfrak{g}'=(\tau',1)\circ J\in\Gamma_{12}'$ such that $\rho_1'(\tau')=\rho_2'(\tau')=g'^2$. We have

$$\begin{aligned}(\delta,\delta)\circ\mathfrak{g}\circ(\delta,\delta)^{-1} &= (\delta,\delta)\circ(\tau,1)\circ J\circ(\delta,\delta)^{-1}=(\delta\tau\delta^{-1},1)\circ J=\\ &= (\delta\tau\delta^{-1}\tau'^{-1},1)\circ(\tau',1)\circ J=(\delta\tau\delta^{-1}\tau'^{-1},1)\circ\mathfrak{g}'\,,\end{aligned}$$

where $(\delta\tau\delta^{-1}\tau'^{-1},1)\in\Gamma_{12}'^0$ because, since $\rho_1'\circ\varphi_\delta=\psi\circ\rho_1$,

$$\rho_1'(\delta\tau\delta^{-1}\tau'^{-1})=\rho_1'(\delta\tau\delta^{-1})\cdot\rho_1'(\tau'^{-1})=\psi(\rho_1(\tau))\cdot g'^{-2}=1\,.$$

This proves that the element (δ,δ) conjugates not only the subgroups Γ_{12}^0 and $\Gamma_{12}'^0$, but the full groups Γ_{12} and Γ_{12}' as well.

Therefore we have the following characterization of isomorphism classes of mixed Beauville surfaces via their defining quadruples.

COROLLARY 5. *Let $q = (G^0; a, b, c)$ and $q' = (G'^0; a', b', c')$ be Beauville structures on G. Then the Beauville surfaces corresponding to q and q' are isomorphic if and only if there exists an automorphism ψ of G with $\psi(G^0) = G'^0$ such that*

$$\psi(a, b, c) \equiv (a', b', c') \bmod \mathrm{I}(G'^0; l, m, n)\,.$$

COROLLARY 6. *The following are invariants of the isomorphism class of a mixed Beauville surface $X = S \times S/G$:*

(i) *the abstract groups G and G^0;*
(ii) *the type (l, m, n) of X;*
(iii) *the twisted isomorphism class of the G^0-covering $S \longrightarrow S/G^0$, hence the Riemann surface S itself.*

5.3. Automorphisms of mixed Beauville surfaces. Proceeding in the same way as in section 4.3, we will study the group of automorphisms of a mixed Beauville surface X. We have the following chain of inclusions

$$\Gamma_{12}^0 \lhd \Gamma_{12} < N(\Gamma_{12}) < N(\Gamma_{12}^0) < \mathrm{Aut}(\mathbb{H} \times \mathbb{H})\,,$$

and the automorphism group of X can be seen as $\mathrm{Aut}(X) \cong N(\Gamma_{12})/\Gamma_{12}$. Consider the intersections

$$\begin{array}{rcl} N_0(\Gamma_{12}) & = & N(\Gamma_{12}) \cap (\mathrm{Aut}(\mathbb{H}) \times \mathrm{Aut}(\mathbb{H})) \qquad \text{and} \\ N_1(\Gamma_{12}) & = & N_0(\Gamma_{12}) \cap (\Gamma \times \Gamma) = N(\Gamma_{12}) \cap (\Gamma \times \Gamma)\,. \end{array}$$

We have a natural isomorphism

$$N_0(\Gamma_{12})/\Gamma_{12}^0 \cong N(\Gamma_{12})/\Gamma_{12} \tag{26}$$

induced by the natural injection of $N_0(\Gamma_{12})$ in $N(\Gamma_{12})$.

As in the unmixed case (section 4.3) we have a homomorphism

$$\begin{array}{rccc} \phi: & N_1(\Gamma_{12}) & \longrightarrow & Z(G^0) \\ & (\gamma_1, \gamma_2) & \longmapsto & \rho_2(\gamma_2)^{-1}\rho_1(\gamma_1) \end{array}$$

whose kernel is Γ_{12}^0.

Choose an element $g \in G \setminus G^0$ and define the subgroup

$$Z(G^0)_{-1} := \{h \in Z(G^0) : gh^{-1}g^{-1} = h\}\,.$$

As any other element of $G \setminus G^0$ is of the form $g' = gh_0$ for some $h_0 \in G^0$, one readily sees that $Z(G^0)_{-1}$ does not depend on the choice of g within the subset $G \setminus G^0$. We claim that $\mathrm{Im}(\phi) = Z(G^0)_{-1}$.

Now recall that a uniformizing group of X was provided by $\Gamma_{12} = \langle \Gamma_{12}^0, \mathfrak{g} \rangle$, where $\mathfrak{g} = (\tau, 1) \circ J$ for any $\tau \in \Gamma$ with $\rho_1(\tau) = g^2$. Therefore any element $(\gamma_1, \gamma_2) \in N(\Gamma_{12})$ must satisfy

$$(\gamma_1, \gamma_2) \circ (\tau, 1) \circ J \circ (\gamma_1, \gamma_2)^{-1} = (\gamma_1 \tau \gamma_2^{-1} \tau^{-1}, \gamma_2 \gamma_1^{-1}) \circ (\tau, 1) \circ J \in \Gamma_{12}\,,$$

i.e. $(\gamma_1 \tau \gamma_2^{-1} \tau^{-1}, \gamma_2 \gamma_1^{-1}) \in \Gamma_{12}^0$, which is equivalent to the equality

$$\rho_2(\gamma_2)\rho_2(\gamma_1)^{-1} = \rho_1(\gamma_1 \tau \gamma_2^{-1} \tau^{-1}) = \rho_1(\gamma_1) g^2 \rho_1(\gamma_2)^{-1} g^{-2}\,. \tag{27}$$

From here a straightforward calculation using the identity (22) gives

$$\phi(\gamma_1, \gamma_2) = g \cdot \phi(\gamma_1, \gamma_2)^{-1} \cdot g^{-1},$$

hence $\phi(\gamma_1, \gamma_2) \in Z(G^0)_{-1}$.

To prove that ϕ is an epimorphism take any $h \in Z(G^0)_{-1}$ and let $\gamma \in \Gamma$ be such that $\rho_1(\gamma) = h$. The element $(\gamma, 1)$ belongs to $N_1(\Gamma_{12})$ since it satisfies formula (27), and clearly $\phi(\gamma, 1) = h$. Therefore we have

$$\frac{N_1(\Gamma_{12})}{\Gamma_{12}^0} \cong Z(G^0)_{-1}\,,$$

which can be regarded as a subgroup of $\mathrm{Aut}(X) = N(\Gamma_{12})/\Gamma_{12}$ via the identification (26).

Now we can prove the following:

Theorem 3. *Let X be a mixed Beauville surface with group G. The group $Z(G^0)_{-1}$ is canonically identified with a subgroup of $\mathrm{Aut}(X)$ of index dividing 36. More precisely, let κ be 6, 2 or 1 depending on whether the type (l, m, n) of X has three, two or no repeated orders. Then there exists a natural number N dividing κ^2 such that*

$$|\mathrm{Aut}(X)| = N \cdot |Z(G^0)_{-1}|\,.$$

In particular, if $\kappa = 1$ then $\mathrm{Aut}(X) \cong Z(G^0)_{-1}$.

Proof. By (26) one has the following equality

$$|\mathrm{Aut}(X)| = |N_0(\Gamma_{12})/\Gamma_{12}^0| = [N_0(\Gamma_{12}) : N_1(\Gamma_{12})] \cdot [N_1(\Gamma_{12}) : \Gamma_{12}^0]\,.$$

Now, by the comments above we have $[N_1(\Gamma_{12}) : \Gamma_{12}^0] = |Z(G^0)_{-1}|$.
On the other hand $N_0(\Gamma_{12}) < N(\Gamma) \times N(\Gamma)$, and so $[N_0(\Gamma_{12}) : N_1(\Gamma_{12})]$ divides $[N(\Gamma) \times N(\Gamma) : \Gamma \times \Gamma] = |N(\Gamma)/\Gamma|^2 = \kappa^2$, and the result follows from (8). □

6. Unmixed Beauville surfaces with group $\mathrm{PSL}(2, p)$ and bitype $((2, 3, n), (p, p, p))$

As an application of the results of section 4.2 we explicitly construct all unmixed Beauville surfaces with group $G = \mathrm{PSL}(2, p)$ and bitype $((2, 3, n), (p, p, p))$, for any prime number $p \geq 13$ and any natural number $n > 6$ dividing either $(p-1)/2$ or $(p+1)/2$. In [**16**] it is proved that for each prime number p the number of isomorphism classes of Beauville surfaces with group $G = \mathrm{PSL}(2, p)$ and given bitype is bounded by a constant that depends on the bitype, but not on p. Here we find that for the particular bitypes we are considering this number is exactly $\phi(n)$ where, as usual, $\phi(n)$ stands for Euler's function.

The next two lemmas describe the number and shape of the triples of generators of types $(2, 3, n)$ and (p, p, p) respectively. The result follows rather easily from basic facts about the group $\mathrm{PSL}(2, p)$ together with results of Macbeath [**23**]. A complete proof can be found in our article [**20**].

Lemma 4. *Let p be a prime number $p \geq 5$ and $n > 6$ any natural number dividing either $(p-1)/2$ or $(p+1)/2$.*

- (i) *There are $\phi(n)$ classes of triples of generators of type $(2, 3, n)$ modulo $\mathrm{I}(G; 2, 3, n) = G$, and $\phi(n)/2$ classes of triples of generators of type $(2, 3, n)$ modulo $\mathrm{A}(G; 2, 3, n) = \mathrm{Aut}(G) \cong \mathrm{PGL}(2, p)$.*
- (ii) *The $\phi(n)/2$ classes modulo $\mathrm{Aut}(G)$ can be represented by triples of the form (a_i, b_i, c^i), where c is an element of order n, a_i and b_i are suitable elements of order 2 and 3, respectively, and $1 \leq i < n/2$ with $\gcd(i, n) = 1$.*

These, together with another set of $\phi(n)/2$ triples (a_i', b_i', c^i) of the same form, provide representatives for the $\phi(n)$ classes modulo G.

(iii) *The conjugacy class of the element c^i of order n characterizes the conjugacy class of the triple modulo* $\mathrm{Aut}(G)$.

(iv) *Corresponding to these triples, there are exactly $\phi(n)/2$ isomorphism classes of triangle $G-$coverings (S_i, f_i) with covering group $G = \mathrm{PSL}(2,p)$ and type* $(2,3,n)$.

By point (ii), any element $\psi \in \mathrm{Aut}(G) \setminus G$ sends the triple (a_i, b_i, c^i) to a triple $\psi(a_i, b_i, c^i)$ which is $G-$equivalent to (a_i', b_i', c^i).

Example 4. For $p = 13$ and $n = 7$ the following triples define the only three triangle $G-$coverings with group $G = \mathrm{PSL}(2,13)$ and type $(2,3,7)$:

$$\begin{aligned} (a_1, b_1, c) &= \left(\begin{pmatrix} 8 & 3 \\ 0 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 8 \\ 8 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 12 & 6 \end{pmatrix} \right), \\ (a_2, b_2, c^2) &= \left(\begin{pmatrix} 0 & 12 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 6 & 12 \\ 4 & 6 \end{pmatrix}, \begin{pmatrix} 12 & 6 \\ 7 & 9 \end{pmatrix} \right), \\ (a_3, b_3, c^3) &= \left(\begin{pmatrix} 12 & 1 \\ 11 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 10 \\ 9 & 1 \end{pmatrix}, \begin{pmatrix} 7 & 9 \\ 4 & 9 \end{pmatrix} \right). \end{aligned}$$

Any other triple (a', b', c') of type $(2,3,7)$ can be mapped by an automorphism of $\mathrm{PSL}(2,7)$ to one of these, depending on the conjugacy class of c'. These three Riemann surfaces are Hurwitz curves of genus 14, i.e. they are Riemann surfaces whose automorphism group attains the Hurwitz bound $|\mathrm{Aut}(S)| \leq 84(g(S) - 1)$.

Lemma 5. *Let $p > 5$ be a prime number.*

(i) *There is only one class of triples of generators of type (p,p,p) modulo $\mathrm{Aut}(G) < \mathrm{A}(G; p,p,p)$, which is represented by*

$$u = \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix}, \quad v = \begin{pmatrix} 3 & -4 \\ 4 & -5 \end{pmatrix}, \quad w = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

(ii) *Corresponding to this triple, there is exactly one isomorphism class of triangle $G-$coverings (S, f) with covering group $G = \mathrm{PSL}(2,p)$ and type (p,p,p).*

(iii) *Modulo $\mathrm{I}(G; p,p,p)$ there are two classes of triples of generators of type (p,p,p), represented by elements of the form (u,v,w) and (u',v',w^ε), where ε is a generator of $\mathbb{F}_p^*$, the group of units of the field with p elements, u, v, w are suitable elements of order p, and $(u',v',w^\varepsilon) = \psi(u,v,w)$ for some $\psi \in \mathrm{Aut}(G) \setminus \mathrm{Inn}(G)$.*

Remark 6. Concerning the point (iii) above we should mention that in [**20**] it is only proved that there are two classes of triples of generators of type (p,p,p) modulo G, not modulo $\mathrm{I}(G; p,p,p)$. However, the given representatives still produce different classes modulo $\mathrm{I}(G; p,p,p)$. In fact, it can be checked that the three elements of the triple u, v and w forming the first triple (resp. u', v' and w^ε forming the second triple) lie on the same conjugacy class of G, while the elements w and w^ε are not conjugate in G (see e.g. [**15**], §5.2). But the action of any element of $\mathrm{I}(G;p,p,p)$ sends (u,v,w) to a triple formed by elements conjugate to u, v and w (see (10)), and so both triples are not equivalent modulo $\mathrm{I}(G;p,p,p)$.

Clearly any pair of triples of generators of G of types $(2,3,n)$ and (p,p,p) satisfy the criterion (14), since the orders are coprime, hence, for any prime number $p>5$ we can introduce the following $\phi(n)$ Beauville surfaces:

- X_i defined by the pairs of triples (a_i,b_i,c^i) and (u,v,w),
- X_i' defined by the pairs of triples (a_i,b_i,c^i) and (u',v',w^ε),

where $\gcd(i,n)=1$ and $i<n/2$. Note that both of them can be written as $S_i \times S/G$, but the action of G on the product $S_i \times S$ is different in each case.

We have the following theorem.

Theorem 4. *Let p be a prime number $p \geq 13$ and $n>6$ any natural number dividing either $(p-1)/2$ or $(p+1)/2$. There are exactly $\phi(n)$ isomorphism classes of Beauville surfaces with group $G=\mathrm{PSL}(2,p)$ and bitype $((2,3,n),(p,p,p))$, represented by the surfaces X_i and X_i' constructed above.*

Proof. By Proposition 5, when defining Beauville surfaces we can consider triples of generators up to the action of $\mathrm{I}(G;l_i,m_i,n_i)$. Therefore the surfaces defined by the following pairs of triples

$$\begin{array}{rclcrcl} t_1(i) &=& (a_i,b_i,c^i\,;\,u,v,w)\ , & & t_2(i) &=& (a_i,b_i,c^i\,;\,u',v',w^\varepsilon)\,, \\ t_1'(i) &=& (a_i',b_i',c^i\,;\,u',v',w^\varepsilon) & \text{and} & t_2'(i) &=& (a_i',b_i',c^i\,;\,u,v,w)\,, \end{array}$$

for $1 \leq i < n/2$ with $\gcd(i,n)=1$ account for all the Beauville surfaces of this type. Note furthermore that each X_i and X_i' are defined by the pairs of triples $t_1(i)$ and $t_2(i)$ respectively.

Now, the pairs of triples $t_1(i)$ and $t_1'(i)$ (resp. $t_2(i)$ and $t_2'(i)$) define the same Beauville surface. In fact, by the two lemmas above any element of $\mathrm{Aut}(G)\backslash G$ sends the triple (a_i,b_i,c^i) to a triple $\mathrm{I}(G;2,3,n)-$equivalent to (a_i',b_i',c^i), and (u,v,w) to a triple $\mathrm{I}(G;p,p,p)-$equivalent to (u',v',w^ε), and the claim follows from Corollary 2.

However, for the same reason $t_1(i)$ and $t_2(i)$ define non-isomorphic Beauville surfaces since, by Corollary 2, this happens if and only if there exists $\psi \in \mathrm{Aut}(G)$ such that

$$\begin{array}{rcl} \psi(a_i,b_i,c^i) &\equiv& (a_i,b_i,c^i) \bmod \mathrm{I}(G;2,3,n) \\ \psi(u',v',w^\varepsilon) &\equiv& (u,v,w) \bmod \mathrm{I}(G;p,p,p) \end{array}$$

simultaneously. Now, the first relation may occur only if $\psi \in G$, and the second one only if $\psi \notin G$.

On the other hand, if $i \neq j$, Corollary 3 implies that the surfaces defined by $t_1(i)$ and $t_2(i)$ and the ones defined by $t_1(j)$ and $t_2(j)$ cannot be isomorphic, since the Riemann surfaces of type $(2,3,n)$ involved in the construction of the first ones are not isomorphic to the ones appearing in the second ones.

Finally, the condition $p \geq 13$ follows from the fact that for prime numbers p with $5<p<13$ there are no natural numbers $n>6$ dividing either $(p-1)/2$ or $(p+1)/2$. □

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