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Genus 2 Belyĭ surfaces with a unicellular uniform dessin

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Abstract A bicoloured graph embedded in a compact oriented surface and dividing it into a union of simply connected components (faces) is known as a dessin d'enfant. It is well known that such a graph determines a complex structure on the underlying topological surface, but a given compact Riemann surface may correspond to different dessins.

In this paper we deal with all unicellular (one-faced) uniform dessins of genus 2 and their underlying Riemann surfaces. A dessin is called uniform if white vertices, black vertices and faces have constant degree, say p, q and r respectively. A uniform dessin d'enfant of type (p, q, r) on a given surface S corresponds to the inclusion of the torsion-free Fuchsian group K uniformizing S inside a triangle group $\Delta(p, q, r)$. Hence the existence of different uniform dessins on S is related to the possible inclusion of K in different triangle groups.

The main result of the paper states that two unicellular uniform dessins belonging to the same genus 2 surface must necessarily be isomorphic or obtained by renormalisation. The problem is approached through the study of the face-centers of the dessins. The displacement of such a point by the elements of K must belong to a prescribed discrete set of (hyperbolic) distances determined by the signature (p, q, r). Therefore looking for face-centers amounts to finding points correctly displaced by every element of K.

Keywords Riemann surfaces · dessins d'enfant · Belyi surfaces · genus 2

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1 Introduction

A *dessin d'enfant* $\mathcal{D}$ is a bipartite graph embedded into an oriented compact surface S and dividing it into topological discs, called the *faces* of the dessin. By work of Grothendieck we know that such a graph defines a Riemann surface structure on S with a morphism to the sphere $\mathbb{P}^1(\mathbb{C})$ with at most 3 ramification values. Such morphisms are called *Belyĭ functions*. A Riemann surface constructed in this way is known as *Belyĭ surface* and it is isomorphic to an algebraic curve defined over a number field by Belyĭ's theorem [4]. To retrieve a dessin from a Belyĭ function β (with normalized ramification values 0, 1 and ∞) one simply takes $\beta^{-1}([0,1])$. The white vertices can be taken to be $\beta^{-1}(0)$ and the black vertices to be $\beta^{-1}(1)$. As for $\beta^{-1}(\infty)$, it consists of one point inside each face, called the *face center*.

The combinatorial structure of a dessin $\mathcal{D}$ is encoded in the so-called *monodromy*, a pair of permutations σ_0 and σ_1 in the symmetric group $\mathfrak{S}_N$, where N is the number of edges. A given cycle of σ_0 (resp. σ_1) describes the edges incident to a particular white vertex (resp. black vertex). Accordingly, $\sigma_\infty = (\sigma_0\sigma_1)^{-1}$ carries information about the faces of $\mathcal{D}$. The group $\langle\sigma_0, \sigma_1\rangle$, called the *monodromy group* of $\mathcal{D}$, is a transitive subgroup of $\mathfrak{S}_N$ since $\mathcal{D}$ is a connected graph.

Conversely, the choice of such a pair of permutations (modulo conjugation inside $\mathfrak{S}_N$) determines a dessin $\mathcal{D}$ (with N edges), up to isomorphism.

Dessins d'enfants arise also from the inclusion of a Fuchsian group K into a triangle group $\Delta = \Delta(p,q,r)$. We can associate to such an inclusion a meromorphic function on the surface $\mathbb{D}/K$, given by the natural projection $\mathbb{D}/K \longrightarrow \mathbb{D}/\Delta \simeq \mathbb{P}^1(\mathbb{C})$, which is a Belyĭ function.

This last point of view allows us to draw a fundamental domain of a Belyĭ surface as union of hyperbolic polygons (forming the faces of the dessin) with certain side-pairings. These polygons consist of several copies of the fundamental domain of the corresponding triangle group $\Delta(p,q,r)$ in which K is included.

In this paper we shall be concerned with dessins consisting of only one face. More precisely, we will deal with the question of whether two dessins drawn on the orientable topological surface of genus 2 and of a naturally defined kind (uniform dessins, see definition below) give rise to isomorphic Riemann surface structures.

2 Multiple uniform dessins of given type on a given surface

A basic problem in the theory of dessins d'enfants comes from the fact that many different dessins may produce the same complex structure in the topological surface where they are embedded. In other words, a given Riemann surface contains many dessins d'enfants. The question of whether or not the Riemann surfaces underlying two given dessins are the same is too difficult to address in its full generality, but restriction to certain classes of dessins can make the question affordable.

The restriction of this problem to *regular dessins* was fully answered in [9]. In terms of Fuchsian groups, the Belyi surface S (often called *quasiplatonic*) underlying a regular dessin $\mathcal{D}$ is uniformized by a normal subgroup K of a Fuchsian triangle group Δ, and the Belyi function corresponding to $\mathcal{D}$ is the obvious map $\mathbb{D}/K \to \mathbb{D}/\Delta \simeq \widehat{\mathbb{C}}$.

The class of dessins where it is more natural to extend the results of [9] is that of *uniform dessins*. They correspond to a uniformizing (i.e. torsion free) group K contained (possibly not normally) inside a triangle group $\Delta = \Delta(p, q, r)$. All the white vertices (resp. black vertices, faces) of the corresponding dessin $\mathcal{D}$ have valency p (resp. valency q, valency $2r$)), and (p, q, r) is said to be the *type* of $\mathcal{D}$.

The existence of multiple dessins of the same type in a given surface $S \simeq \mathbb{D}/K$ can be described in two ways in terms of the related Fuchsian groups:

1. There exists some $\alpha \in \mathrm{PSL}_2(\mathbb{R})$ such that $K < \Delta$ and $K < \alpha\Delta\alpha^{-1}$.
2. There exists some $\alpha \in \mathrm{PSL}_2(\mathbb{R})$ such that $K < \Delta$ and $\alpha^{-1}K\alpha < \Delta$.

Both descriptions look equivalent, and they are so apart from two special situations. When α belongs to $N(K)$, the normalizer of K in $\mathrm{PSL}_2(\mathbb{R})$, conjugation by α corresponds to an element of the automorphism group of the surface $\mathrm{Aut}(S) \simeq N(K)/K$. Conversely if two isomorphic uniform dessins belong to a given surface S, then there exists an automorphism of S that moves one to the other. In this situation only the first description shows the existence of two different dessins group theoretically.

The second special situation occurs for $\alpha \in N(\Delta)$, hence it fits better with the second description. In this case, conjugation by α corresponds to some permutation of the role of white vertices, black vertices and face centers, and the dessin given by the inclusion $\alpha^{-1}K\alpha < \Delta$ is a renormalisation of the one corresponding to $K < \Delta$. This situation can only happen for signatures where two of the integers p, q, r coincide, since otherwise $N(\Delta) = \Delta$.

As a consequence, note that two different inclusions $K < \Delta$ and $K' < \Delta'$ determine isomorphic or renormalised dessins $\mathcal{D}$ and $\mathcal{D}'$ if and only if there exists $\alpha \in \mathrm{PSL}_2(\mathbb{R})$ conjugating the whole inclusion, that is

$$K' = \alpha K \alpha^{-1} \qquad \text{and} \qquad \Delta' = \alpha\Delta\alpha^{-1}.$$

Remark 1 Note that two uniform dessins of different types (p_1, q_1, r_1) and (p_2, q_2, r_2) may lie on the same surface, but then the associated triangle groups $\Delta(p_1, q_1, r_1)$ and $\Delta(p_2, q_2, r_2)$ must be commensurable, that is their intersection is a subgroup of finite index in both of them.

When dealing with these inclusions, we will use some general facts about triangle groups, so we recall them next.

For every hyperbolic signature (p, q, r) there is a unique (modulo conjugation in $\mathrm{PSL}_2(\mathbb{R})$) triangle group $\Delta = \Delta(p, q, r)$ of this type. It is generated by elliptic elements γ_0, γ_1, and γ_∞ of order p, q and r, which are rotations around the vertices of a hyperbolic triangle T with angles π/p, π/q and π/r respectively. A fundamental domain for Δ can be obtained as the union of T with its mirror image across any chosen side.

The hyperbolic distances between the fixpoints of any two elements in the conjugacy class of γ_∞ inside Δ, form a discrete set $d(p, q, r) = \{d_1 < d_2 < \cdots\}$ that does not depend on the choice of the triangle group within its conjugacy class. We will call $d(p, q, r)$ the set of *admissible distances* for the type (p, q, r). These hyperbolic distances were already used in [2] and [7], and will play a special role in our study.

Now let $\mathcal{D}$ be a uniform dessin given by $K < \Delta(p,q,r)$, where $|\Delta(p,q,r) : K| = N$. A fundamental domain for K will consist of $N/r = s$ hyperbolic polygons with centers at $c_1, \dots, c_s$, all of them fixpoints of elements in the conjugacy class of γ_∞ inside Δ; each polygon corresponds to one of the faces of $\mathcal{D}$. Without loss of generality we can assume c_1 to be the origin.

Note also that, since $K < \Delta$, every element of K sends each c_i to another point of order r in the tessellation. In particular every such an element of K moves c_i by one of the admissible distances listed in $d(p,q,r)$.

Suppose we have two different uniform dessins $\mathcal{D}$ and $\mathcal{D}'$ of type (p,q,r) in the same surface $S = \mathbb{D}/K$, given by the inclusions $K < \Delta$ and $K < \alpha\Delta\alpha^{-1}$. For each of these dessins we have a fundamental domain for K as before, with face centers $0 = c_1, \dots, c_s$ and $\alpha(0) = c'_1, c'_2, \dots, c'_s$ respectively. Now c'_i $(i = 1, \dots, s)$ are order r vertices of the tessellation of $\alpha\Delta\alpha^{-1}$, but they are not necessarily vertices of the tessellation of Δ. However, the condition of being moved an admissible distance by every element of K applies to both c_i and c'_i, thus we have the following:

Lemma 1 *Let $\mathcal{D}$ be a dessin given by the inclusion $K < \Delta(p,q,r)$, and let $\mathcal{D}'$ be another dessin of the same type in the same surface. If $z \in \mathbb{D}$ corresponds to a face center of $\mathcal{D}'$, then for every $g \in K$*

$$\rho(z, g(z)) \in d(p,q,r),$$

where ρ stands for the hyperbolic distance.

On the other hand, given $K < \Delta(p,q,r)$, any point of $\mathbb{D}$ moved an admissible distance by every transformation in K will be called an *admissible point*, since it is a candidate for being the face center of a uniform dessin of type (p,q,r) on the surface $\mathbb{D}/K$.

The following facts about hyperbolic geometry ([3], Ch. 7) will be useful for our purposes:

Lemma 2 *Let g be a hyperbolic isometry of the disc and let us define the* translation length *of g as $T_g = \inf \rho(z, g(z))$. Then:*

1. *The* axis *of g, i.e. the set of points translated precisely T_g by g, is the geodesic joining the two fixpoints of g. We will denote it by A_g.*
2. *The hyperbolic distance between a point z and its image under g depends only on the distance from z to the axis. More precisely*

$$\sinh \frac{\rho(z, g(z))}{2} = \cosh \rho(z, A_g) \sinh \frac{T_g}{2}.$$

3. *The set of points $C_d(g)$ of $\mathbb{D}$ that is moved a given distance $d > T_g$ by g forms two arcs of (generalized) circles passing through the fixpoints of g.*

By the convexity of the hyperbolic distance and the previous lemma, the displacement of the points in a hyperbolic triangle is bounded from above by the displacement of its vertices [7], that is:

Lemma 3 *Let T be a hyperbolic triangle with vertices at v_1, v_2, v_3, and let g be a hyperbolic isometry. Then*

$$\rho(z, g(z)) \leq \max\{\rho(v_1, g(v_1)), \rho(v_2, g(v_2)), \rho(v_3, g(v_3))\}$$

for all $z \in T$.

Under certain conditions that will be clear in each case, the inequality in Lemma 3 is strict for all $z \in T \setminus \{v_1, v_2, v_3\}$.

3 Unicellular uniform dessins of genus 2

We shall deal from now on with uniform dessins with only one face (*unicellular dessins*) on surfaces of genus $g = 2$. For general results on general uniform dessins in arbitrary genus, see [8].

The list of triangle signatures of uniform dessins of genus 2 can be found in [16]. For each of them there is a given index $[\Delta(p, q, r) : K]$ and a number N of non-isomorphic dessins of type (p, q, r), or equivalently the number of conjugacy classes of genus 2 subgroups $K < \Delta(p, q, r)$. In Table 1 we show the signatures corresponding to unicellular dessins. We note in passing that all the triangle groups involved here are arithmetic [17]. For an account of the role played by arithmeticity in the problems under consideration see [8].

Signature	Index	N
(5,5,5)	5	4
(3,6,6)	6	4
(2,8,8)	8	4
(3,3,9)	9	4
(2,5,10)	10	7
(2,4,12)	12	6
(2,3,18)	18	9

Table 1 Unicellular uniform dessins in genus 2.

In terms of the monodromy, since by definition of uniform dessin the cycle structure of σ_0, σ_1 and $\sigma_\infty = (\sigma_0\sigma_1)^{-1}$ is determined by the signature, finding all non-isomorphic dessins of type (p, q, r) amounts to finding all the permutations σ_0, σ_1 and $\sigma_0\sigma_1$ with the given structure, modulo conjugation in $\mathfrak{S}_m$, the symmetric group on $m = [\Delta(p, q, r) : K]$ elements. They can be easily computed with help of any standard algebraic software such as GAP or Magma.

When a uniform dessin $\mathcal{D}$ is unicellular one of the orders of the generators of the monodromy coincides with the index of K inside the triangle group. Under these assumptions σ_∞ can be chosen to be a single cycle of maximal length, and so $\mathcal{D}$ has only one face. Equivalently, $\beta^{-1}(\infty)$ consists of only one point in the surface, where β is the Belyĭ function associated to $\mathcal{D}$.

All the triangle groups appearing in Table 1 belong to different commensurability classes except for $\Delta(5, 5, 5) < \Delta(2, 5, 10)$ and $\Delta(3, 3, 9) < \Delta(2, 3, 18)$ (see [17]).

Therefore by Remark 1 two dessins of different such signatures will certainly belong to non isomorphic Riemann surfaces, except perhaps for the exceptional ones.

The results obtained in the next sections have the following consequence.

Theorem 1 *Two unicellular uniform dessins of the same type in genus 2 belong to the same surface if and only if they are either isomorphic or obtained by renormalisation.*

We shall stress here the fact that this statement is known to be false if we remove the condition on the number of faces. The surface S_{23} below contains two uniform dessins of type $(2, 3, 9)$ (not unicellular). The same phenomenon occurs in higher genus: Klein's surface of genus 3 contains two uniform dessins of type $(2, 3, 7)$ (see [8]). On the other hand, to our knowledge it is still unknown whether or not the statement remains true for unicellular dessins in arbitrary genus.

The proofs of the results we present here are based on the following method. Let $S = \mathbb{D}/K$ be the genus 2 surface underlying a certain unicellular dessin $\mathcal{D}$ of type (p, q, r). We consider the corresponding polygon P of $2r$ sides and angles $2\pi/p$ and $2\pi/q$ that is a fundamental domain for K. Face centers of uniform (p, q, r)-dessins on S must be detectable as admissible points. In order to find them we look for points in P that are moved an admissible distance by (at least) all the side-pairings generating K. Discerning the true face centers among these admissible points requires further arguments, mainly on the automorphisms of S.

The search of points mentioned above is computer aided, and the results will be presented through graphics produced with Mathematica [18] and the package [11]. Most times the results are visualized very clearly in the figures. In the few cases when there could be any doubt about some point, due to the lack of precision of the graphic, we will provide a precise argument.

Note that finding two different unicellular dessins amounts to find their face centers, since there cannot be two dessins centered at the same point. To see this suppose we have two dessins on $S \simeq \mathbb{D}/K$ centered at $[0]_K$, the point of S corresponding to $0 \in \mathbb{D}$, given by inclusions $K < \Delta$ and $K < \alpha\Delta\alpha^{-1}$. In particular α can be chosen to be a rotation around the origin, since both triangle groups must have an order-r fixpoint at 0. Furthermore, it can be seen that α must preserve the set of fixpoints of order r of Δ, and so it preserves the fundamental polygon of K as well. Note that in all the cases we consider here our fundamental polygon agrees with the Dirichlet fundamental region around the center of the polygon, hence the position of the centers of all the polygons of the tessellation, which are the order-r fixpoints, determines the shape and position of our polygon, and therefore $\alpha\Delta\alpha^{-1} = \Delta$ and the two dessins are the same.

Figure 1 shows the kind of elements that will appear in the pictures. We label the edges and the vertices z_i of the fundamental polygon P counterclockwise. The i-th edge joins z_{i-1} and z_i, and the edge 1 is the one intersecting $\mathbb{R}^+$. We denote as well p_i as the hyperbolic midpoint of the i-th edge. The notation will be slightly different for the particular case $(p, q, r) = (3, 6, 6)$, see Section 3.5.

In all the figures, $F^-_{(i,j)}$ and $F^+_{(i,j)}$ denote the repelling and attracting fixpoint of $\gamma_{(i,j)}$, the transformation that sends the i-th edge of P to the j-th one. The arcs joining these two points represent admissible arcs for $\gamma_{(i,j)}$, and a label d_k will be placed on $C_{d_k}(\gamma_{(i,j)})$, where d_k is the k-th admissible distance (recall the notation from Lemma 2).

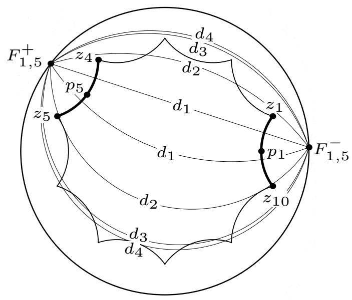


Fig. 1 Some of the admissible arcs for the side-pairing $\gamma_{(1,5)}$ in the particular case $(p, q, r) = (5, 5, 5)$

The next sections contain the discussion for each possible signature. Every section begins with a (graphic) enumeration of all the different dessins, shown as side-pairings on the fundamental polygon. In order to obtain all these side-pairings we have previously computed all possible monodromies of the dessins involved.

3.1 Dessins of type (5,5,5)

The 4 uniform dessins of type $(5, 5, 5)$ are displayed in Figure 2, where lines indicate the side-pairings.

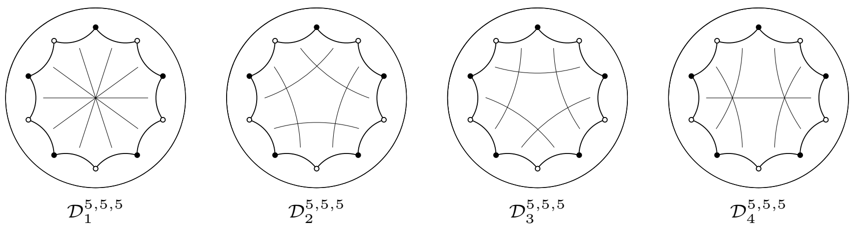


Fig. 2 The uniform dessins of type $(5, 5, 5)$

Black vertices, white vertices and face centers can be interchanged to obtain the renormalised graphs, which are still of type $(5, 5, 5)$. The dessins $\mathcal{D}_2$ and $\mathcal{D}_3$ are obviously obtained by renormalisation, since one is obtained from the other by interchanging the vertices colours. In fact $\mathcal{D}_1$ is also a renormalised dessin, since changing in it the face center with white or black vertices gives $\mathcal{D}_2$ or $\mathcal{D}_3$ (Figure 3).

On the contrary $\mathcal{D}_4^{5,5,5}$ is self-dual, i.e. isomorphic to its renormalisations.

Theorem 2 *There are two uniform Belyĭ surfaces of type* $(5, 5, 5)$*.*
The following table shows the uniform $(5, 5, 5)$*-dessins contained on them along with the location of their face centers.*

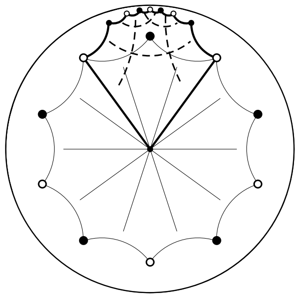


Fig. 3 $\mathcal{D}_2^{5,5,5}$ and $\mathcal{D}_3^{5,5,5}$ are renormalisations of $\mathcal{D}_1^{5,5,5}$

Surface	*Aut(S)*	*Dessins*	*Centers*
S_1	C_{10}	$\mathcal{D}_1$	$[0]$
		$\mathcal{D}_2,\ \mathcal{D}_3$	$[z_3],\ [z_8]$
S_2	$\mathfrak{S}_3 \times C_2$	$\mathcal{D}_4$	$[0]$, $[z_3]$, $[z_8]$, $[p_1]$, $[q]$, $[-q]$, where $q = \sqrt{-2+\sqrt{5}\, i}$

Proof The rotation of order 10 around the origin induces an automorphism σ_{10} in S_1, the surface underlying $\mathcal{D}_1$. Note that the order 2 automorphism σ_{10}^5 fixes 6 points, and therefore it corresponds to the hyperelliptic involution J. By the well known list of possible automorphisms group of genus 2 in [14] we find that $\mathrm{Aut}(S_1)$ must be a cyclic group of order 10 and that $y^2 = x^5 - 1$ is an algebraic equation for S_1. It follows that there does not exist a second dessin on S_1 isomorphic to $\mathcal{D}_1$, since otherwise more automorphisms would exist on the surface.

In [6] the authors give the equation $y^2 = x^6 + 118x^3/5 + 1$ for the surface S_2 underlying $\mathcal{D}_4$. The rotation of order 2 around the origin induces now an automorphism $\sigma_2 \neq J$ that interchanges $[z_3]$ and $[z_8]$ (black and white vertices in $\mathcal{D}_4$). Since this dessin is self-dual there is also an order 2 automorphism fixing $[z_3]$ (resp. $[z_8]$), that lifts to an order 2 rotation around, say, z_3 (resp. z_8) and interchanging the points $[0]$ and $[z_8]$ (resp. $[0]$ and $[z_3]$).

These three automorphisms generate a subgroup of $\mathrm{Aut}(S_2)$ not containing the hyperelliptic involution J. Since this subgroup can be seen as all the permutations of $[0]$, $[z_3]$ and $[z_8]$, it is clearly isomorphic to $\mathfrak{S}_3$, the symmetric group on three elements.

Note that σ_2 fixes two points, namely $[0]$ and $[p_1]$. Since every automorphism commutes with the hyperelliptic involution J, we have

$$\sigma_2 \circ J([0]) = J \circ \sigma_2([0]) = J([0]), \qquad \text{and} \qquad \sigma_2 \circ J([p_1]) = J \circ \sigma_2([p_1]) = J([p_1]).$$

It follows that $J([p_1])$ and $J([0])$ are fixed points of σ_2. Since $J([0]) \neq [0]$ we must have $J([0]) = [p_1]$, therefore there must be a second dessin isomorphic to $\mathcal{D}_4$ centered at $[p_1]$.

We can proceed in the same way with the automorphism fixing $[z_3]$ (resp. $[z_8]$) whose second fixed point is $[-q]$ (resp. $[q]$), the hyperbolic midpoint of $[0, z_8]$ (resp. the hyperbolic midpoint of $[0, z_3]$). Therefore we find two dessins isomorphic to $\mathcal{D}_4$ with the face center at $[-q] = J([z_3])$ and $[q] = J([z_8])$.

If there were more dessins isomorphic to $\mathcal{D}_4$ in S_2 the automorphism group would be larger than $\mathfrak{S}_3 \times C_2$. By [14] the only possibilities are either $D_6 \times C_2$ or $\mathfrak{S}_4 \times C_2$, and they correspond to the surfaces S_7 and S_{17} (see Propositions 4 and 6).

Finally, surfaces S_1 and S_2 are not isomorphic because they have non isomorphic automorphism group, therefore they only contain the dessins we have already found above.

3.2 Dessins of type (2,5,10)

The 7 uniform dessins of type $(2, 5, 10)$ are given by the side-pairings of a decagon shown in Figure 4. Let us remark that here $p = 2$, $q = 5$ and $r = 10$, and we can think of the fundamental polygon as a decagon with angle $2\pi/5$ instead of a 20-gon with half the angles equal to π.

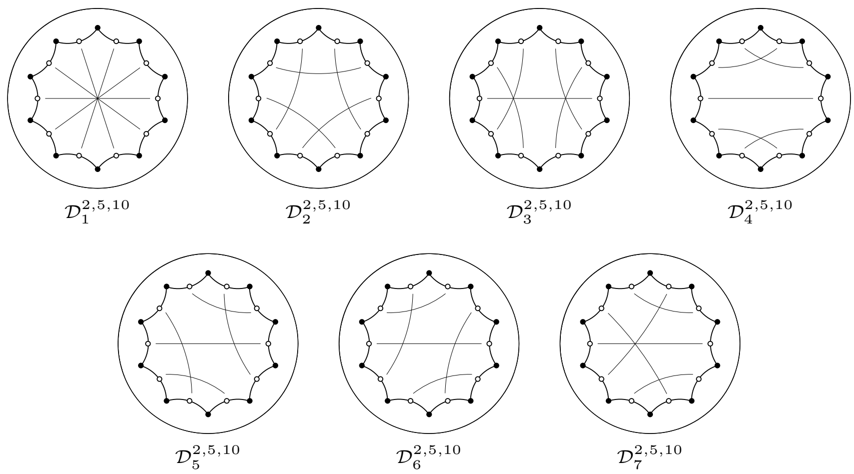


Fig. 4 Dessins of type $(2, 5, 10)$

Note first that any surface containing a $(5, 5, 5)$-dessin contains as well a $(2, 5, 10)$-dessin, which is obtained by refinement of the former one. This is due to the inclusion of triangle groups $\Delta(5, 5, 5) < \Delta(2, 5, 10)$. Note also that Proposition 2 produces an equivalent statement regarding $(2, 5, 10)$-dessins on S_1 and S_2.

With this in mind it is obvious that this refinement procedure transforms $\mathcal{D}_1^{5,5,5}$ into $\mathcal{D}_1^{2,5,10}$, $\mathcal{D}_2^{5,5,5}$ and $\mathcal{D}_3^{5,5,5}$ into $\mathcal{D}_2^{2,5,10}$, and $\mathcal{D}_4^{5,5,5}$ into $\mathcal{D}_3^{2,5,10}$. In particular, since $\mathcal{D}_1^{5,5,5}$, $\mathcal{D}_2^{5,5,5}$ and $\mathcal{D}_3^{5,5,5}$ were renormalisations of each other, it follows that $\mathcal{D}_1^{2,5,10}$ and $\mathcal{D}_2^{2,5,10}$ belong to the same surface.

Theorem 3 *There are six uniform Belyĭ surfaces of type* $(2, 5, 10)$*:*

Surface	*Aut*(S)	*Dessins*	*Centers*
S_1	C_{10}	$\mathcal{D}_1$ $\mathcal{D}_2$	$[0]$ $[z_3]$, $[z_8]$
S_2	$\mathfrak{S}_3 \times C_2$	$\mathcal{D}_3$	$[0]$, $[z_3]$, $[z_8]$, $[p_1]$, $[q]$, $[-q]$, where $q = \sqrt{-2+\sqrt{5}\,i}$
S_3	$C_2 \times C_2$	$\mathcal{D}_4$	$[0]$, $[p_1]$
S_4	$C_2 \times C_2$	$\mathcal{D}_5$	$[0]$, $[p_1]$
S_5	$C_2 \times C_2$	$\mathcal{D}_6$	$[0]$, $[p_1]$
S_6	C_2	$\mathcal{D}_7$	$[0]$, $[q]$, where $q = -\sqrt{-\frac{1}{2}+\frac{3\sqrt{5}}{10}}$

Proof Let S_3 be the surface underlying $\mathcal{D}_4$. The order two rotation around the origin induces an automorphism σ_2 of the surface fixing 0 and p_1. This automorphism being different from the hyperelliptic involution J we can proceed as in the proof of Theorem 2 to deduce that $J([0]) = [p_1]$. The same argument applies for the surfaces S_4 and S_5 underlying $\mathcal{D}_5$ and $\mathcal{D}_6$.
We know then that in S_3, S_4 and S_5 there is a second dessin isomorphic to the original one centered in $[p_1]$.

In order to prove that these three surfaces are non isomorphic and that they do not contain further $(2, 5, 10)$-dessins we look now for other admissible points.

In the case of S_3, the two symmetries given by reflection over the imaginary and real axes allow us to study just one quarter of the fundamental domain, say the upper-right one. The map $\gamma_{(2,4)}$ translates the points p_2, z_2 and z_3 strictly less than d_1 (see Figure 5 where the two components of $C_{d_1}(\gamma_{(2,4)})$ are displayed), so by Lemma 3 there can be no admissible point in the convex sub-polygon generated by such points and the origin (the shaded region in the figure). The only points in the remaining region which are translated an admissible distance by $\gamma_{(2,4)}$ and by $\gamma_{(1,6)}$ simultaneously are the origin, p_1 and some point x in the real axis. But the latter one is translated by $\gamma_{(3,5)}$ a non-admissible distance $d_1 < \rho(x, \gamma_{(3,5)}(x)) < d_2$. Therefore, besides $[0]$ only $[p_1]$ can be the face center of a $(2, 5, 10)$-dessin.

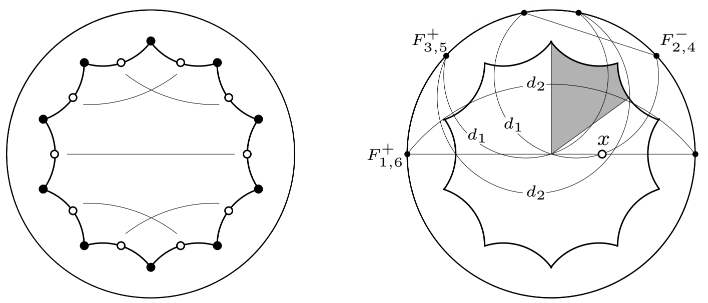


Fig. 5 After discarding the point x, the only admissible points left for $\mathcal{D}_4$ are the origin and p_1

Let S_4 and S_5 be the surfaces corresponding to $\mathcal{D}_5$ and $\mathcal{D}_6$ respectively. The study of both surfaces is essentially the same, since they are related by an obvious anti-conformal involution of the disc (in particular the corresponding algebraic curves are complex conjugate).
By the symmetry of $\mathcal{D}_5$ we can focus just on the upper half part. As before $\gamma_{(2,4)}$ translates the points p_2, z_2 and z_3 strictly less than d_1, as also does with p_4. We can discard the hyperbolic sub-polygon with these points plus the origin as vertices. On the other hand $\gamma_{(5,8)}$ translates p_5 less than d_1, and exactly d_1 the points z_5 and p_6. Therefore, we can get rid also of the sub-polygon with vertices at 0, p_5, z_5 and p_6 with the only exception of the points p_6, which must be a face center since $[p_6] = [p_1]$, and z_5 (that will be discarded later).

The remaining regions to be considered are the quadrilaterals R_1 and R_2 with vertices at p_1, z_1, p_2, 0 and p_4, z_4, p_5, 0 respectively (see Figure 6). The only points of R_1 which are admissible for $\gamma_{(1,6)}$ and $\gamma_{(2,4)}$ simultaneously are the origin, p_1, and the same point x in the study of $\mathcal{D}_4$, which is translated less than d_1 by $\gamma_{(3,10)}$. Note that by getting rid of z_1 we discard z_5 as well since they correspond to the same point on the surface.

Looking at the intersections of admissible arcs for $\gamma_{(1,6)}$ and $\gamma_{(5,8)}$ in R_2, we get two candidates. One of them is z_4 (not relevant since it is identified with z_1), and the other one is some point y in the interior of the polygon (see Figure 6), which can be discarded by looking at the side-pairing $\gamma_{(2,4)}$.

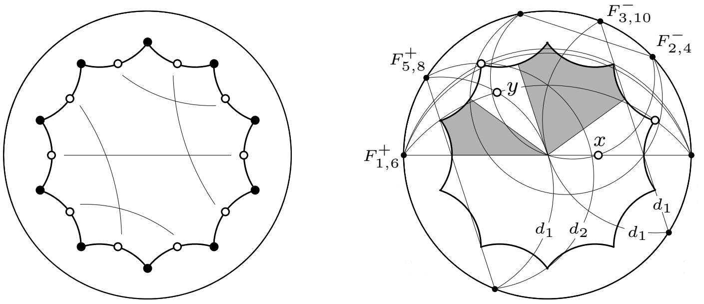


Fig. 6 Only $[p_1]$ is a new face center in S_4

Finally, let S_6 be the surface underlying $\mathcal{D}_7$. We can study just half of the polygon due to the (orientation reversing) symmetry of the identification pattern. The side-pairing $\gamma_{(2,4)}$ allows us to dispose of the same region as before. The region still not considered is the union of the sub-polygon R_1 with vertices p_4, z_4, z_5, p_6 and 0, and the quadrilateral R_2 with vertices p_1, z_1, p_2, 0 (see Figure 7).

In R_1 the admissible arcs for $\gamma_{(3,7)}$ intersect those of $\gamma_{(1,6)}$ at three points: the origin, q and y. The point y is clearly translated a non admissible distance between d_1 and d_2 by $\gamma_{(2,4)}$ (see Figure 7), hence it can be discarded. The point q is the intersection of $C_{d_1}(\gamma_{(1,6)})$, $C_{d_2}(\gamma_{(2,4)})$, $C_{d_1}(\gamma_{(3,7)})$, $C_{d_1}(\gamma_{(5,9)})$ and $C_{d_2}(\gamma_{(8,10)})$.

In R_2 the only points with admissible displacement by $\gamma_{(1,6)}$ and $\gamma_{(2,4)}$ are once again p_1 and x. The latter is translated a distance in between d_1 and d_2 by $\gamma_{(3,7)}$.

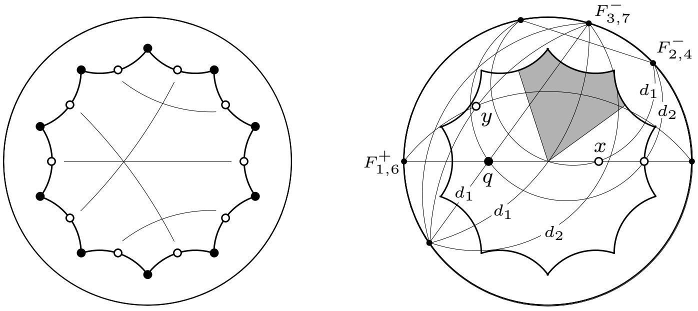


Fig. 7 In S_6 the point $[q]$ is the only face center apart from $[0]$

In order to get rid of the point p_1 it is enough to note that the equivalent point p_6 has been already discarded.

Finally, since the order 2 rotation around the center does not induce the hyperelliptic involution, a dessin isomorphic to $\mathcal{D}_7$ must be found centered elsewhere, and $[q]$ is the only possibility.

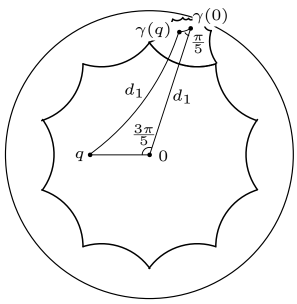


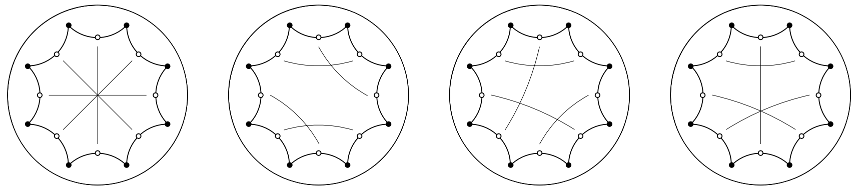
Fig. 8 Explicit calculation of q. Here γ stands for $\gamma_{(7,3)}$

The exact coordinates of q are determined by its hyperbolic distance to the origin. This distance can be explicitly found using hyperbolic trigonometry in the quadrilateral with vertices 0, q, $\gamma_{(7,3)}(q)$ and $\gamma_{(7,3)}(0)$ (see Figure 8). The calculations, lengthy but fairly simple, consist of consecutive applications of the sine and cosine rules ([3], p.148), and lead us to the exact algebraic value $q = -\sqrt{-\frac{1}{2}+\frac{3\sqrt{5}}{10}}$.

3.3 Dessins of type (2,8,8)

The 4 uniform dessins of type $(2,8,8)$ are shown in Figure 9.

These dessins were already considered in [1]. In this case interchanging the role of the black vertices and face centers produces renormalised dessins again of type



$\mathcal{D}_1^{2,8,8}$ $\mathcal{D}_2^{2,8,8}$ $\mathcal{D}_3^{2,8,8}$ $\mathcal{D}_4^{2,8,8}$

Fig. 9 Dessins of type $(2, 8, 8)$

$(2, 8, 8)$. However one can easily check that this process does not relate any two of these dessins, since they are all self-dual.

Theorem 4 *There are four uniform Belyĭ surfaces of type* $(2, 8, 8)$*:*

Surface	*Aut*(S)	*Dessins*	*Centers*
S_7	$\mathfrak{S}_4 \times C_2$	$\mathcal{D}_1$	$[0]$, $[z_1]$, $[p_j]$, with $j = 1, \dots 4$
S_8	$C_2 \times C_2$	$\mathcal{D}_2$	$[0]$, $[z_1]$
S_9	$C_2 \times C_2$	$\mathcal{D}_3$	$[0]$, $[z_1]$, $[q_1]$, $[q_2]$, where: $q_1 = \sqrt{-4 + 3\sqrt{2}}\, e^{\pi i/8}$ and $q_2 = \frac{\sqrt[4]{2}}{2} e^{9\pi i/8}$
S_{10}	$C_2 \times C_2$	$\mathcal{D}_4$	$[0]$, $[z_1]$, $[q_1]$, $[q_2]$, where: $q_1 = \frac{\sqrt[4]{2}}{2}\sqrt{2 + \sqrt{2}}\left(1 - \frac{\sqrt{2}}{3} + \frac{i}{3}\right)$ and $q_2 = -\frac{\sqrt{-2+2\sqrt{2}}}{2} i$

Proof The rotation of order 8 around the origin induces an automorphism σ_8 in S_7, the surface associated to $\mathcal{D}_1$. It can be seen that σ_8^4 corresponds to the hyperelliptic involution, whose fixed points are $[0]$, $[z_1]$ and $[p_j]$, with $j = 1, \dots, 4$. Note that in particular, the face center of any dessin isomorphic to $\mathcal{D}_1$ is a Weierstrass point. Now, according to [14] the only possibility for $\mathrm{Aut}(S_7)/J$ to have an element of order four is the symmetric group $\mathfrak{S}_4$, and S_7 corresponds to the algebraic curve $y^2 = x(x^4 - 1)$. The automorphism group of this curve acts transitively on the set of Weierstrass points, therefore we have automorphisms sending $[0]$ to $[p_j]$ for $j = 1, \dots, 4$. In particular, these points are face centers of dessins isomorphic to $\mathcal{D}_1$.

The fact that any $(2, 8, 8)$-dessin non isomorphic to $\mathcal{D}_1$ cannot belong to S_7 will follow from the study of the other dessins.

Let S_8 be the surface associated to $\mathcal{D}_2$. We already know that the renormalised dessin centered at $[z_1]$ is isomorphic to $\mathcal{D}_2$. We will see that there are no more face centers.

By symmetry we can reduce our study of admissible points to the quarter of the polygon with, say, vertices z_2, z_3, z_4 and 0. The identification $\gamma_{(2,4)}$ translates p_2 and p_4 strictly less than d_1, and the points z_2 and z_3 exactly the first admissible distance. In fact in the sub-polygon with these points together with the origin as vertices all the points are translated strictly less than d_1, except for the origin, z_2

and z_3 (these last two represent the same point in the surface). The only remaining region is the triangle with vertices at the origin, p_4 and z_4 (see Figure 10).

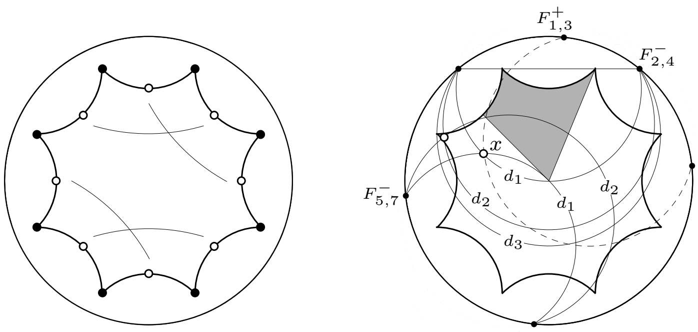


Fig. 10 Some admissible arcs mentioned in the study of S_8

In this triangle there are two common admissible points for $\gamma_{(2,4)}$ and $\gamma_{(5,7)}$ (the white points in Figure 10). Both of them are not admissible for $\gamma_{(1,3)}$, but they lie so close to some of its admissible arcs that this time it does not result transparent in the figure. For instance, the dashed arc in Figure 10 representing $C_{d_2}(\gamma_{(1,3)})$, does not pass through the point x at the intersection of $C_{d_1}(\gamma_{(2,4)})$ and $C_{d_1}(\gamma_{(5,7)})$, although it looks so in the picture. This can be seen through an explicit computation as follows.

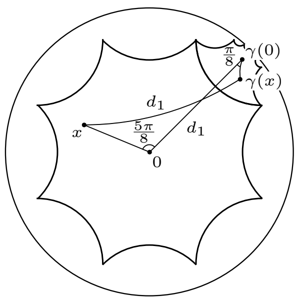


Fig. 11 The coordinates of x are computed through hyperbolic trigonometry

First we compute x with help of hyperbolic trigonometry (Figure 11). Using that x and 0 are the only points in $[0, z_4]$ translated d_1 by $\gamma_{(4,2)}$, we find that $x = \sqrt{-4 + 3\sqrt{2}}\, e^{7\pi i/8}$. Finally, we check that x is moved by $\gamma_{(1,3)}$ a non admissible distance $d \approx 4.253$ between $d_2 \approx 4.218$ and $d_3 \approx 4.741$. A similar argument discards the other white point in Figure 10.

In S_9, the surface associated to $\mathcal{D}_3$ it is enough to study one half of the polygon, say the one with vertices $z_1, z_2, \ldots, z_5$. Using the same argument as before, $\gamma_{(2,4)}$ discards the grey region in Figure 12, except for the points z_2 and z_3. Now there are two regions left to be studied: the triangle R_1 with vertices z_1, p_2 and 0, and the quadrilateral R_2 with vertices p_4, z_4, z_5 and 0.

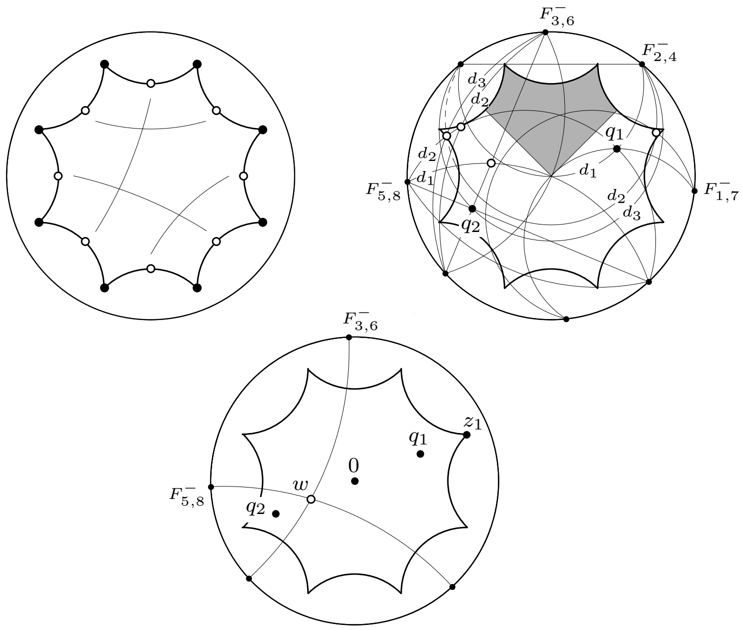


Fig. 12 Only q_1 and q_2 are admissible points. The surface S_9 contains dessins isomorphic to $\mathcal{D}_3$ centered at $[z_1]$, $[q_1]$ and $[q_2]$

There are two common admissible points for $\gamma_{(2,4)}$ and $\gamma_{(1,7)}$ in the region R_1, apart from 0 and z_1 (already face centers). Numerical computations show again that one of them, the white point in Figure 12, is not admissible for $\gamma_{(5,8)}$. Now the same calculation we did for the point x in S_8 gives the value $q_1 = \sqrt{-4+3\sqrt{2}}\, e^{\pi i/8}$ for the second one.

In R_2 we focus on the common admissible points for $\gamma_{(3,6)}$ and $\gamma_{(5,8)}$. There are four points apart from 0, z_4 and z_5. Two of them are easily discarded because they are moved a non admissible distance between d_1 and d_2 by $\gamma_{(2,4)}$ (see Figure 12). The two points left are q_2 lying in the segment $[0, z_5]$ and some point x in the intersection of $C_{d_3}(\gamma_{(3,6)})$ and $C_{d_2}(\gamma_{(5,8)})$ near z_4.
Now x is translated a non admissible distance slightly greater than d_2, but it is again not possible to see this in the figure (see the dashed arc). An argument like the previous ones can be made, though. The exact computation of q_2 is very similar to the one for the point q in S_6, and gives $q_2 = \frac{\sqrt[4]{2}}{2} e^{9\pi i/8}$.

The reason why $[q_1]$ and $[q_2]$ are indeed face centers of dessins isomorphic to $\mathcal{D}_3$ is that the hyperelliptic involution J maps $[0]$ to $[q_2]$ and $[z_1]$ to $[q_1]$. To see this, we argue as follows. The axes of $\gamma_{(3,6)}$ and $\gamma_{(5,8)}$ intersect at a point w in the segment $[0, q_2]$. Now $[w] \in S_9$ is the intersection of the two simple closed geodesics determined by the two axes, and according to [10] it must be a Weierstrass point.

Moreover, J lifts to an order 2 rotation around w, which is the (hyperbolic) midpoint of $[0, q_2]$, and we have $J([0]) = [q_2]$ and $J([z_1]) = [q_1]$.

In S_{10}, the surface underlying $\mathcal{D}_4$, it is once again enough to study only one half of the polygon, say the right half. The side-pairing $\gamma_{(2,4)}$ allows us to discard the grey region in Figure 13. We will divide the study of the remaining region in two parts: the part with positive imaginary part and the one with negative one.

In the first one there are three points which are admissible both for $\gamma_{(2,4)}$ and $\gamma_{(1,6)}$. Two of them are discarded by the side-pairing $\gamma_{(3,7)}$ (see the dashed arcs in Figure 13). The only point left in this region is some point q_1 lying in the edge number two of the octagon. We shall determine its value later.

For the second region left to study, we look for common admissible points for $\gamma_{(5,8)}$ and $\gamma_{(1,6)}$. There are three such points, and the only one which is admissible for $\gamma_{(3,7)}$ is q_2 lying on the imaginary line. The computation of q_2, based on the fact that it is the only point in the imaginary line apart from 0 moved d_1 by $\gamma_{(1,6)}$, follows the same lines as the computation of q in the study of S_6. The result is $q_2 = -\frac{\sqrt{-2+2\sqrt{2}}}{2} i$.

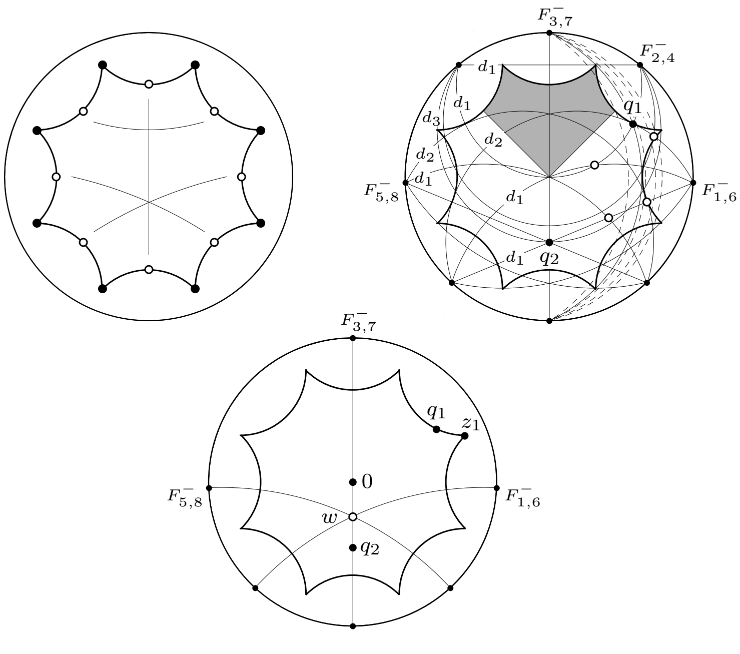


Fig. 13 We find four admissible points in S_{10}, namely 0, z_1, q_1 and q_2. There is a dessin isomorphic to $\mathcal{D}_4$ centered at each one of them

Once more the intersection of axes of side-pairings are not the midpoints between the center and any vertex, so there must be additional admissible points, the images under the hyperelliptic involution J of $[0]$ and $[z_1]$, and they are precisely $[q_1]$ and $[q_2]$. In the right part of Figure 13 we represent a Weierstrass point w in the intersection of the axes of $\gamma_{(1,6)}$, $\gamma_{(3,7)}$ and $\gamma_{(5,8)})$. The order 2 rotation around w is a lift of J mapping 0 to q_2 and z_1 to a point identified with q_1.

We can compute now q_1 easily. Since we know that there is a uniform $(2,8,8)$-dessin isomorphic to $\mathcal{D}_4$ centered at q_1, there must be an admissible point lying at the same (hyperbolic) distance from q_1 as the (hyperbolic) distance d from q_1 to the origin. It seems that this can be nothing else than the vertex v_1, so we claim that the length of the segment of the second side joining v_1 to q_1 equals precisely d.

With some easy hyperbolic trigonometry, and the help of Maple, our claim would yield the value

$$q_1 = \frac{\sqrt[4]{2}}{2}\sqrt{2+\sqrt{2}}\left(1-\frac{\sqrt{2}}{3}+\frac{i}{3}\right).$$

Finally, it is quite straightforward to check that q_1 is moved d_1 by $\gamma_{(2,4)}$, d_2 by $\gamma_{(1,6)}$, and d_3 by $\gamma_{(3,7)}$, as expected (see Figure 13). The above value of q_1 is therefore correct.

3.4 Dessins of type (2,4,12)

The 6 uniform dessins of type $(2,4,12)$ correspond to the side-pairings of a dodecagon shown in Figure 14.

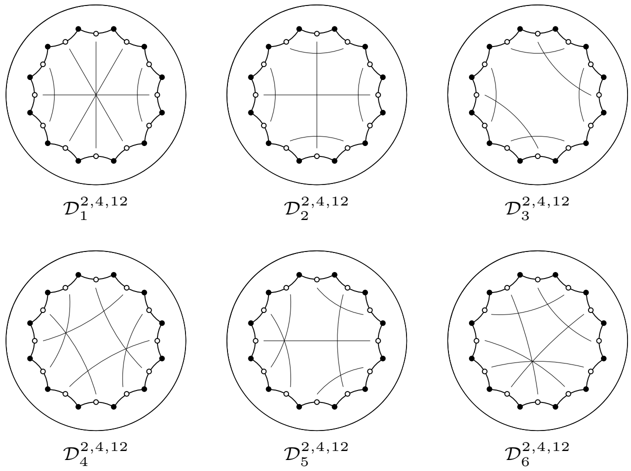


Fig. 14 Dessins of type $(2,4,12)$

Theorem 5 *There are six uniform Belyĭ surfaces of type* $(2,4,12)$*:*

Surface	*Aut(S)*	*Dessins*	*Centers*
S_{11}	C_2	$\mathcal{D}_1$	[0]
S_{12}	$C_2 \times C_2$	$\mathcal{D}_2$	[0], $[z_2]$
S_{13}	$C_2 \times C_2$	$\mathcal{D}_3$	[0], $[z_2]$
S_{14}	$C_2 \times C_2$	$\mathcal{D}_4$	[0], $[z_3]$
S_{15}	C_2	$\mathcal{D}_5$	[0], $[q]$, where $q = \frac{\sqrt{6}}{6}\sqrt[4]{3}$
S_{16}	C_2	$\mathcal{D}_6$	[0], $[q]$, where $q = \sqrt{-2 + \frac{4}{3}\sqrt{3}}\, e^{17\pi i/12}$

Proof Let us note first that the hyperelliptic involution on the surface S_{11} (related to the dessin $\mathcal{D}_1$) lifts to a rotation around the origin of the disc, therefore it preserves the dessin $\mathcal{D}_1$. By the symmetries of the side-pairings we can study only one quarter of the domain, say the upper-right one. The only points in the shaded region depicted in Figure 15 that are translated an admissible distance by the transformation $\gamma_{(2,12)}$ are the origin and z_2. Now, the arc $C_{d_1}(\gamma_{(4,10)})$, which is the axis of $\gamma_{(4,10)}$, coincides with the imaginary line. In addition this identification moves strictly less than d_2 the vertices z_3 and z_4. Using again Lemma 3 and considering the identification $\gamma_{(3,9)}$ as well, we can get rid of all the upper-right quarter of the domain, with the exception of the segments $[0, p_3]$ and $[0, p_4]$. There are two common admissible points for $\gamma_{(3,9)}$ and $\gamma_{(4,10)}$ (represented as white dots in Figure 15), but none of them is translated an admissible distance by $\gamma_{(2,12)}$.

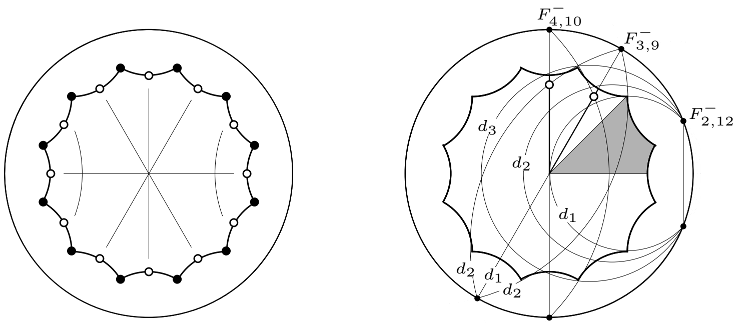


Fig. 15 S_{11} does not contain any other dessin apart from $\mathcal{D}_1$

For S_{12}, the surface underlying $\mathcal{D}_2$, it is enough to study one quarter of the domain as well. Notice that the hyperelliptic involution J does not fix [0] so there must be at least another admissible point. The transformation $\gamma_{(3,5)}$ moves the points z_2 and z_5 exactly the first admissible distance d_1, and the points p_3, z_3, p_4, z_4 and p_5 strictly less than d_1, so we can exclude the interior of the convex sub-polygon generated by these vertices and the origin. In fact the only points translated a distance d_1 in this sub-polygon are the origin, z_2 and z_5 (see Figure 16). By the symmetries of the side-pairings we have four admissible points z_2, z_5, z_8 and z_{11} (apart from the origin), all of them corresponding to the same point $[z_2]$ on the surface. Since J does not fix [0] there must be at least another center, and so $[z_2]$ is the center of a dessin isomorphic to $\mathcal{D}_2$, image of the original one under J.

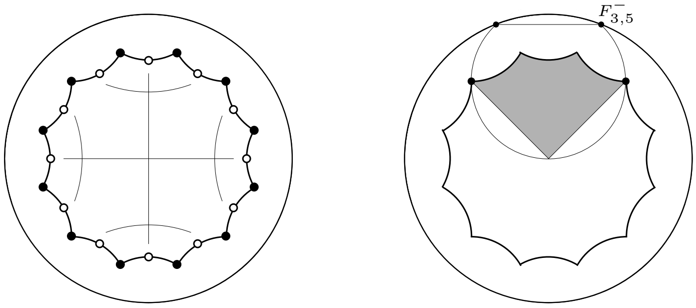


Fig. 16 The vertices z_2 and z_5 correspond to the same point on S_{12}, which is the center of a dessin

The same argument can be applied to S_{13}, the surface underlying $\mathcal{D}_3$, since the four side-pairings used for S_{12} are side-pairings of this surface too. The only points not discarded are once again $[0]$ and $[z_2]$, that are necessarily related by the hyperelliptic involution.

Let S_{14} be the surface corresponding to $\mathcal{D}_4$. We can focus on one quarter of the domain, say the one between z_{12} and z_3. The side-pairings $\gamma_{(2,11)}$ and $\gamma_{(3,7)}$ discard the grey regions in Figure 17, except for z_{12} and z_3 which are moved a distance d_1 by $\gamma_{(2,11)}$ and $\gamma_{(3,7)}$ respectively. Only one point of the remaining region (see Figure 17) is admissible for both identifications. This point is moved by $\gamma_{(5,8)}$ a non admissible distance in between d_2 and d_3. In Figure 17 there is also an admissible point for both $\gamma_{(2,11)}$ and $\gamma_{(3,7)}$ near the edge number 2, but it lies in fact outside the polygon and, in any case, it is not admissible for $\gamma_{(5,8)}$.

Therefore the only admissible points in the whole polygon are z_{12}, z_3, z_6 and z_9, all of them corresponding to the same point on the surface. Since the rotation of order 2 around the origin does not induce the hyperelliptic involution, this point must be the face center of a dessin isomorphic to $\mathcal{D}_4$.

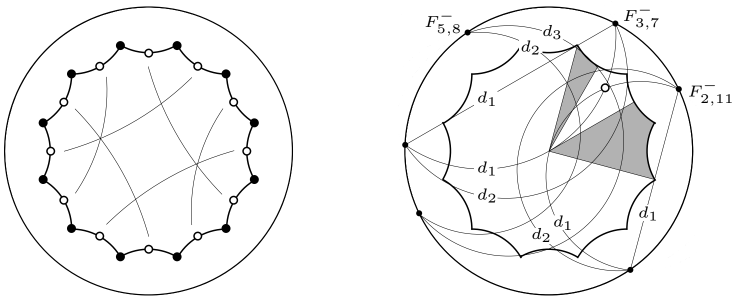


Fig. 17 S_{14} contains only $\mathcal{D}_4$ and its image under J

The surface S_{15} underlying the dessin $\mathcal{D}_5$ has a symmetry of order 2 over the real line. Focusing on the upper half of the domain, the identifications $\gamma_{(2,4)}$ and $\gamma_{(5,8)}$ allow us to get rid of the grey regions in Figure 18, except for z_1, z_4 and

z_6, and they all correspond to the same point on the surface. But z_4, say, can be discarded by looking at $C_{d_1}(\gamma_{(5,8)})$ and $C_{d_2}(\gamma_{(5,8)})$.

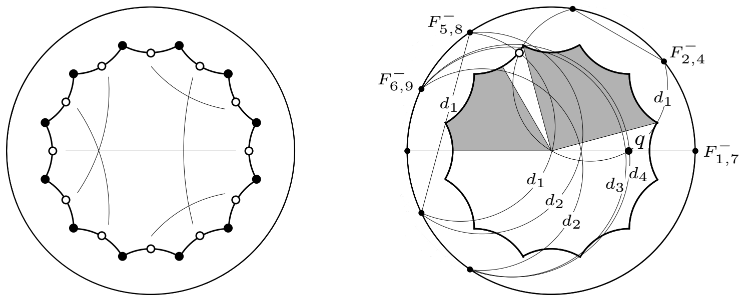


Fig. 18 In S_{15} there is a dessin isomorphic to $\mathcal{D}_5$ centered in $[q]$

Now we only have to deal with the triangles with vertices at z_4, p_5, 0, and p_1, z_1, 0. In the first one, the only common admissible point for both $\gamma_{(2,4)}$ and $\gamma_{(5,8)}$ is the one depicted in white in Figure 18, that is moved a distance between d_2 and d_3 by $\gamma_{(6,9)}$. In the second one there is just one admissible point for $\gamma_{(2,4)}$ and $\gamma_{(1,7)}$, call it q (which is the intersection of $C_{d_1}(\gamma_{(1,7)})$, $C_{d_1}(\gamma_{(2,4)})$ and $C_{d_4}(\gamma_{(6,9)})$). We can compute its value $q = \frac{\sqrt{6}}{6}\sqrt[4]{3}$ with the same methods as for q_1 in S_9, relying on the fact that q is translated a distance d_1 by $\gamma_{(2,4)}$.

Since the hyperelliptic involution J does not fix $[0]$, the point $[q]$ must be the face center of a dessin isomorphic to $\mathcal{D}_3$.

Finally, let S_{16} be the surface corresponding to the dessin $\mathcal{D}_6$. Because of its symmetry we will study only the half polygon on the right of the segment $[z_3, z_9]$. The side-pairing $\gamma_{(1,4)}$ moves p_1, z_1, z_2 and z_3 less than d_1, so we can get rid of the sub-polygon with these and the origin as vertices. In addition, $\gamma_{(7,11)}$ (resp. $\gamma_{(8,12)}$) translates p_{11} (resp. p_{12}) less than the first admissible distance and z_{10} (resp. z_{11}) exactly d_1. Now z_{10} and z_{11} are non admissible for $\gamma_{(8,12)}$ and $\gamma_{(7,11)}$ respectively. As a consequence we can discard all the grey regions in Figure 19.

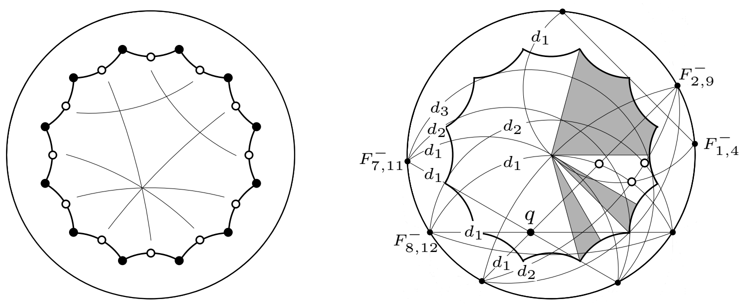


Fig. 19 S_{16} contains a dessin isomorphic to $\mathcal{D}_6$ centered at $[q]$

We focus now on the common admissible points of $\gamma_{(8,12)}$ and $\gamma_{(2,9)}$ in the region left to study. These are the point q depicted in black in Figure 19 and the three white points. These last three can be discarded by looking at the admissible arcs for $\gamma_{(7,11)}$. Hence $[q]$, the only admissible point left, must be the face center of a dessin, image of the original dessin $\mathcal{D}_6$ under J. To determine the coordinates of q we look for the point in the segment $[0, z_9]$ which is moved d_1 by $\gamma_{(9,2)}$, and arguing as for q in S_6 it follows that $q = \sqrt{-2 + \frac{4}{3}\sqrt{3}}\, e^{17\pi i/12}$.

3.5 Dessins of type (3,6,6)

In this case the fundamental polygon is not regular, in contrast to previous cases. The fundamental polygon is now an irregular dodecagon with vertices of angle $2\pi/3$ and $2\pi/6$. The 4 uniform dessins of type $(3, 6, 6)$ are given by the side-pairings shown in Figure 20.

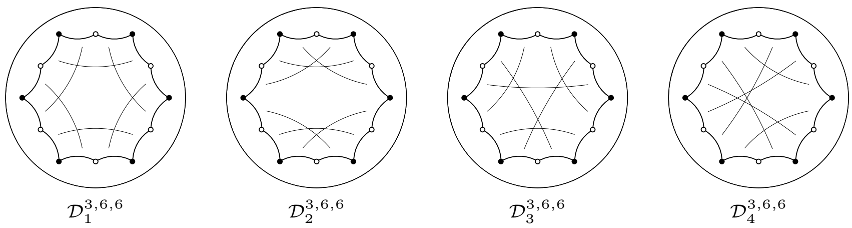


Fig. 20 Dessins of type $(3, 6, 6)$

We denote now by z_i, for $i = 1, \ldots, 12$, the vertices of the polygon, where $z_{12} \in \mathbb{R}^+$ and they are numbered counterclockwise. The i-th edge will be the one joining z_{i-1} and z_i .

As in the case of the $(2, 8, 8)$ dessins, we can switch the role of the black vertices and face centers to get renormalisations of the same type $(3, 6, 6)$. It can be easily seen that all of them are self-dual.

Theorem 6 *There are four uniform Belyĭ surfaces of type* $(3, 6, 6)$*:*

Surface	*Aut(S)*	*Dessins*	*Centers*
S_{17}	$D_6 \times C_2$	$\mathcal{D}_1$	$[0]$, $[z_1]$, $[z_2]$, $[z_3]$
S_{18}	$C_2 \times C_2$	$\mathcal{D}_2$	$[0]$, $[z_2]$
S_{19}	$\mathfrak{S}_3 \times C_2$	$\mathcal{D}_3$	$[0]$, $[z_1]$, $[z_2]$, $[z_3]$
S_{20}	$C_2 \times C_2$	$\mathcal{D}_4$	$[0]$, $[z_2]$, $[q_1]$, $[q_2]$, where $q_1 = \frac{\sqrt{6}}{4}$ and $q_2 = -\frac{\sqrt{6}}{6}$

Proof Let S_{17} be the surface associated to $\mathcal{D}_1$. The rotation of order 6 around the origin induces an automorphism σ_6 of the surface fixing $[0]$ and $[z_2]$, the other vertex of order 6. Note that the order 2 automorphism σ_6^3 does not correspond to the hyperelliptic involution, so $\mathrm{Aut}(S_{17})/J$ contains an order 6 subgroup isomorphic to C_6. By the table on [14] the only possibility for S_{17} is to have an automorphism group isomorphic to $D_6 \times C_2$ and to be isomorphic to the algebraic curve $y^2 = x^6 - 1$.

The side-pairing $\gamma_{(3,12)}$ translates the origin and the consecutive vertices z_{11}, z_{12}, z_1, z_2 and z_3 exactly d_1. In fact these are the only points translated d_1 in the sub-polygon formed by those vertices and the origin (see Figure 21), which is already one third of the whole fundamental domain. Taking into account the symmetry of the picture and considering side-pairings we conclude that the only possible admissible points in the surface are [0], $[z_1]$, $[z_2]$ and $[z_3]$.

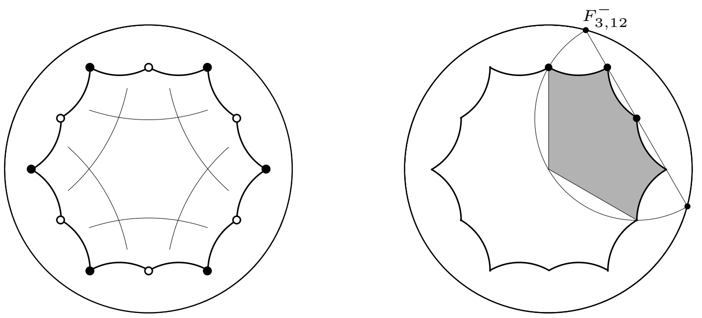


Fig. 21 In the study of $\mathcal{D}_1$ the arcs forming $C_{d_1}(\gamma_{(2,5)})$ discard one third of the domain, except for z_1, z_2 and z_3

By renormalisation we already know that $[z_2]$ is the center of a dessin isomorphic to $\mathcal{D}_1$. To see that the other two admissible points are face centers of a dessin isomorphic to $\mathcal{D}_1$ as well, we can simply draw a suitable fundamental domain around both of them, and check that the side-pairings are the same as the ones for $\mathcal{D}_1$.

In the surface S_{18} corresponding to $\mathcal{D}_2$ there is an obvious symmetry that allows us to study just the upper-right quarter of the domain. The side-pairing $\gamma_{(1,4)}$ translates z_1, and z_3 less than d_1, while z_2 is moved exactly the first distance. Therefore we can discard the grey region of Figure 22, except for z_2 which is a face center as we already know by renormalisation. Apart from z_{12}, there is only one point in the region left to study which is admissible for both $\gamma_{(1,4)}$ and $\gamma_{(9,12)}$. This point, that lies on the real line, is translated a non admissible distance in between d_4 and d_5 by $\gamma_{(3,6)}$, so it can be discarded. Therefore in S_{18} there are only two dessins, namely $\mathcal{D}_2$ and its renormalised dessin.

Let S_{19} be the surface underlying $\mathcal{D}_3$. Using the same argument as for S_{17} we can conclude that there are three other dessins isomorphic to $\mathcal{D}_3$ centered at $[z_1]$, $[z_2]$ and $[z_3]$.

As for S_{20} the surface underlying $\mathcal{D}_4$ once again the symmetry of the side-pairings allows us to study only the upper half of the fundamental polygon. The same argument on $\gamma_{(1,4)}$ and $\gamma_{(9,12)}$ already used for S_{18} allows us to get rid of the upper-right quarter of the domain except for the vertices z_{12} and z_2, and the

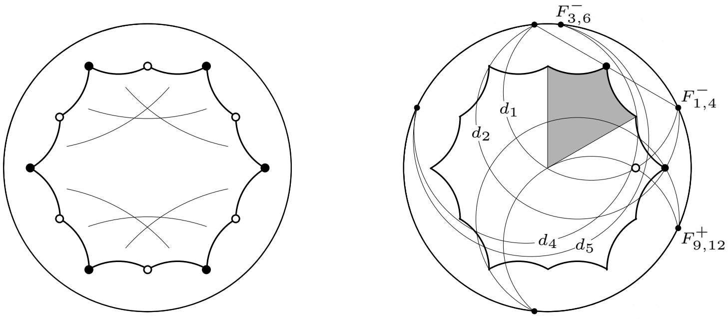


Fig. 22 Only $\mathcal{D}_2$ and its renormalised dessin live in S_{18}

point $q_1 \in \mathbb{R}$ (see Figure 23). Through an argument similar to the one for q_1 in S_9 we find $q_1 = \frac{\sqrt{6}}{4}$.

This last point is now translated an admissible distance by the rest of side-pairings, so it is an admissible point.

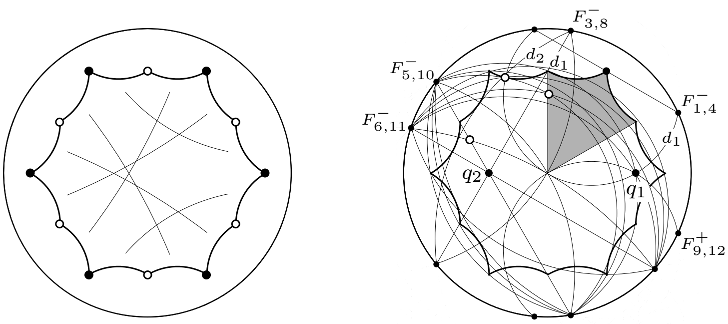


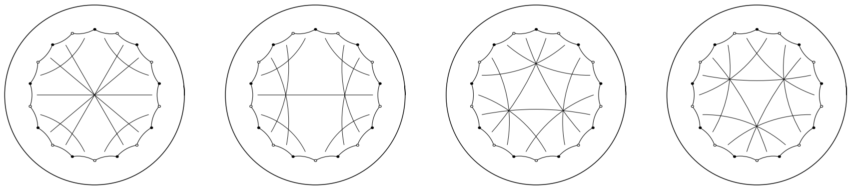
Fig. 23 The only additional points not discarded in the study of $\mathcal{D}_4$ are z_2, q_1 and q_2

For the study of the upper-left quarter we focus on the common admissible points for both $\gamma_{(5,10)}$ and $\gamma_{(6,11)}$. There are three such points apart from the origin, z_4 and z_6, although only $q_2 \in \mathbb{R}$ is translated an admissible distance by $\gamma_{(3,8)}$, and it is in fact another admissible point. Proceeding as before with an argument similar to the one used for the point q in S_6 we find that $q_2 = -\frac{\sqrt{6}}{6}$.

Now the axes of $\gamma_{(2,7)}$, $\gamma_{(3,8)}$, $\gamma_{(5,10)}$ and $\gamma_{(6,11)}$ intersect at w, the hyperbolic midpoint of the segment $[0, q_2]$. Following once again [10] the point $[w]$ of S_{20} must be a Weierstrass point and hence $J([0]) = [q_2]$ and $J([z_2]) = [q_1]$.

3.6 Dessins of type (3,3,9) and of type (2,3,18)

Finally, the results for the types $(3, 3, 9)$ and $(2, 3, 18)$ follow from the study of extremal discs in genus 2 performed in [7].
The 4 uniform dessins of type $(3, 3, 9)$ are displayed in the following figure:



$\mathcal{D}_1^{3,3,9}$ $\mathcal{D}_2^{3,3,9}$ $\mathcal{D}_3^{3,3,9}$ $\mathcal{D}_4^{3,3,9}$

Fig. 24 Dessins of type $(3,3,9)$

The uniform dessins of type $(2,3,18)$ were already studied in [7] in the context of extremal discs, and the details about them can be found there. We know that all of them lie in different surfaces, and the same stands for $\mathcal{D}_1^{3,3,9}$ and $\mathcal{D}_2^{3,3,9}$ (that produce the obvious dessins of type $(2,3,18)$ by refinement).

The other $(3,3,9)$-dessins $\mathcal{D}_3^{3,3,9}$ and $\mathcal{D}_4^{3,3,9}$ are related by renormalisation, and therefore they determine the same surface. Let us remark that, as shown in [8] this last surface has equation $y^2 = x^6+8x^3+4$ and contains two non-isomorphic uniform dessins of type $(2,3,9)$.

Theorem 7 *There are 3 uniform Belyĭ surfaces of type* $(3,3,9)$*:*

Surface	$Aut(S)$	*Dessins*	*Centers*
S_{21}	C_2	$\mathcal{D}_1$	$[0]$
S_{22}	$C_2 \times C_2$	$\mathcal{D}_2$	$[0],\ [p_1]$
S_{23}	$\mathfrak{S}_3 \times C_2$	$\mathcal{D}_3,\ \mathcal{D}_4$	$[0],\ [z_1],\ [z_3],\ [z_5]$

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