$\mathbb{Z}_N$-curves possessing no Thomae formulae of Bershadsky-Radul type

Gabino González-Diez David Torres-Teigell

Abstract

A $\mathbb{Z}_N$-curve is one of the form $y^N = (x-\lambda_1)^{m_1} \dots (x-\lambda_s)^{m_s}$. When $N = 2$ these curves are called hyperelliptic and for them Thomae proved his classical formulae linking the theta functions corresponding to their period matrices to the *branching values* $\lambda_1, \dots, \lambda_s$. In his work on Fermionic fields on $\mathbb{Z}_N$-curves with arbitrary N, Bershadsky and Radul discovered the existence of generalized Thomae's formulae for these curves which they wrote down explicitly in the case in which all *rotation numbers* m_i equal 1. This work was continued by several authors and new Thomae's type formulae for $\mathbb{Z}_N$-curves with other rotation numbers m_i were found. In this article we prove that for some choices of the rotation numbers the corresponding $\mathbb{Z}_N$-curves do not admit such generalized Thomae's formulae.

Mathematics Subject Classification 2010. 14H51, 14K25, 32G81.

Keywords. Riemann surfaces · Special divisors · Thomae type formulae.

Notation. *Throughout this paper we use the following notation. Given an integer* $n \in \mathbb{Z}$ *we shall denote by* $\overline{n} \in \{0, 1, \dots, p-1\}$ *its remainder modulo* p. *By the floor and the ceiling of an arbitrary real number* $x \in \mathbb{R}$ *to be denoted* $\lfloor x \rfloor$ *and* $\lceil x \rceil$ *respectively we will refer to the integers* $\lfloor x \rfloor = \max\{z \in \mathbb{Z} \ : \ z \leq x\}$ *and* $\lceil x \rceil = \min\{z \in \mathbb{Z} \ : \ z \geq x\}$.

1 Introduction

A compact Riemann surface S is called *cyclic N-gonal* if it possesses an automorphism τ of order N such that the quotient $S/\langle\tau\rangle$ has genus zero in which case the natural map $S \to S/\langle\tau\rangle \simeq \widehat{\mathbb{C}}$ provides a degree N morphism that ramifies at the points fixed by τ. Accordingly, the set of fixed points will be referred to as *the ramification (or branch) locus*, and its image in $\widehat{\mathbb{C}}$ as *the set of branching values.*

This paper was first published in Lett. Math. Phys. 98 (2011), no. 2, 193-205. The final publication is available at www.springerlink.com

It is well known (see e.g. [10]) that such a Riemann surface can be seen as as a $\mathbb{Z}_N$-curve, that is S is isomorphic to the Riemann surface of an algebraic curve of the form

$$y^N = (x-\lambda_1)^{m_1} \dots (x-\lambda_s)^{m_s} \tag{1.1}$$

where τ can be viewed as the automorphism $\tau(x,y) = (x, e^{2\pi i/N}y)$ and the set of branching values is $\mathcal{B} = \{\lambda_1, \dots, \lambda_s\}$.

Let g be the genus of S and let D denote a degree g integral divisor, that is $D = P_1^{d_1} \cdots P_\ell^{d_\ell}$ with $P_i \in S$, $d_i \geq 0$ and $\sum d_i = g$. Recall that D is said to be *special* if there is a non constant function f whose set of poles is bounded by D. The significance of the special divisors can be explained as follows. Let $\{a_i, b_i\}_{i=1}^g$ be a symplectic basis of $H_1(S, \mathbb{Z})$, $\{\omega_i\}_{i=1}^g$ the corresponding dual basis of holomorphic 1-forms, $\Omega = (\int_{b_j} \omega_i)$ its period matrix and $J(S) = \mathbb{C}^g/\mathbb{Z}^g \bigoplus \mathbb{Z}^g\Omega$ its jacobian. If we identify the set of integral divisors of degree g with the g-fold symmetric product $S^{(g)}$ then, after choosing a base point $Q \in S$, one has a holomorphic map, *the Abel-Jacobi map*, from $S^{(g)}$ to $J(S)$ defined by

$$A(D) = \sum_{i=1}^{s} d_i \int_Q^{P_i} (\omega_1, \dots, \omega_g) \in J(S)$$

It is a classical result that this is a surjective birational map which fails to be an isomorphism precisely at the special divisors.

An alternative interpretation of this fact comes from Riemann's vanishing theorem. The Riemann theta function is given by the formula

$$\theta(z, \Omega) = \sum_{n \in \mathbb{Z}^g} e^{2\pi i (\frac{1}{2} n^t \cdot \Omega \cdot n + n \cdot z)}$$

This a function defined on $\mathbb{C}^g$ whose zero set is well defined on $J(S)$. Classically to each $\epsilon, \delta \in \mathbb{Q}^g$ one associates the *theta characteristic*

$$\epsilon + \delta \cdot \Omega \in \mathbb{Q}^g \oplus \mathbb{Q}^g \cdot \Omega$$

and the *theta constant*

$$\theta \begin{bmatrix} \epsilon \\ \delta \end{bmatrix} := e^{2\pi i (\frac{1}{2}\epsilon \cdot \Omega \epsilon + \epsilon \cdot \delta)} \theta(\epsilon + \delta \cdot \Omega, \Omega)$$

Now, there is a constant $k_Q \in J(S)$, *the Riemann's constant*, such that if $\epsilon + \delta \cdot \Omega = k_Q + A(D)$ with D special then $\theta \begin{bmatrix} \epsilon \\ \delta \end{bmatrix} = 0$.

We refer to [7] and [13] for background in this subject.

Theta characteristics of the form $\epsilon + \delta \cdot \Omega = k_Q + A(D)$ with D special are called singular theta characteristics. For a hyperelliptic (or $\mathbb{Z}_2$) curve of genus g with equation $y^2 = (x - \lambda_1) \dots (x - \lambda_{2g+1})$ *Thomae's formulae* (see [15], [16], or [9], [13], [5] for a modern version) express the values of the theta constants corresponding to the one-half non singular theta characteristics $\epsilon + \delta \cdot \Omega \in \frac{1}{2}\mathbb{Z}^g \bigoplus \frac{1}{2}\mathbb{Z}^g\Omega$ in terms of the coefficients λ_j. It turns out that all such theta characteristics correspond to non-special divisors D supported on the branch locus.

In his work on Fermionic fields on arbitrary $\mathbb{Z}_N$-curves Bershadsky and Radul [3], building on work by Belavin and Knizhnik [1] and themselves [2], discovered formulae which generalise Thomae's formulae in the sense that they express the theta constants corresponding to the $\frac{1}{2N}$-th non singular theta characteristics $\epsilon + \delta \cdot \Omega = k_Q + A(D) \in \frac{1}{2N}\mathbb{Z}^g \bigoplus \frac{1}{2N}\mathbb{Z}^g\Omega$, where D is a non-special divisor supported on the branch locus, in terms of the parameters λ_j. They succeeded to get an explicit description of them in the case in which s is a multiple of N and $m_1 = \ldots = m_s = 1$. Their proof relies on the path integral formulation of the conformal field theory. Some years later Nakayashiki [14] provided a proof of these same formulae by means of more classical mathematical methods. Afterwards Enolski and Grava [6] achieved the same goal for the case $s = 2d$ and $m_1 = \ldots = m_d = N - 1$, $m_{d+1} = \ldots = m_{2d} = 1$.

It seems to be generally believed that analogous formulae can be found in arbitrary $\mathbb{Z}_N$-curves (see e.g. [3] § 6, [6] § 7 or indeed the erroneous Lemma 2.3 in the first author's paper [10], later corrected in [11]).

In this paper we show that this is not the case

Theorem 1. *For any set of branching values $\mathcal{B} = \{\lambda_1, \ldots, \lambda_s\} \subset \widehat{\mathbb{C}}$ with $s \geq 3$, there are $\mathbb{Z}_N$-curves*

$$y^N = (x - \lambda_1)^{m_1} \ldots (x - \lambda_s)^{m_s}$$

with N arbitrary large possessing no non-special integral divisors of degree g supported on the branch locus.

The proof of this theorem will be a direct consequence of Propositions 1 and 2 in Section 3, where explicit families of curves satisfying the conclusion above are constructed.

Finally, in view of the current activity in the search of Thomae's formulae (see e.g. [4], [5], [12], and above all the book [8]), in Section 4 we give complete lists of all non-special integral divisors (supported on the branch locus) for some $\mathbb{Z}_N$-curves with small values of the degree N and of the number of branching values s.

2 Characterization of non-special integral divisors supported on the branch locus

For the sake of simplicity from now on we shall assume that $N = p$ is a prime number so that our $\mathbb{Z}_N$-curves can be written in the form

$$y^p = (x - \lambda_1)^{m_1} \ldots (x - \lambda_{r+2})^{m_{r+2}} \tag{2.1}$$

where

- $\sum m_i = np$, for some positive integer n, and
- $1 \leq m_i \leq p - 1$
- $r \geq 1$ since we want the genus to be greater than 1.

In this case our $\mathbb{Z}_N$-curves enjoy the following properties

- The associated Riemann surface S consists of the affine points of the curve of equation 2.1 plus p points at infinity.
- The cyclic group $\langle \tau \rangle$ is generated by the automorphism $\tau(x, y) = (x, e^{2\pi i/p}y)$.
- The full fixed point set of τ is $\text{Fix}(\tau) = \{Q_1 = (\lambda_1, 0), \ldots, Q_{r+2} = (\lambda_{r+2}, 0)\}$. The points at infinity get permuted by τ.
- The rotation angle of τ^{-1} at a fixed point Q_k is $e^{2\pi i m_k/p}$ (that is, locally $\tau^{-1}(z) = e^{2\pi i m_k/p} \cdot z$).

- The genus of S is $g = \frac{p-1}{2} r$.

It is a trivial fact that if one of the exponents in our degree g divisor $D = Q_1^{d_1} \cdots Q_{r+2}^{d_{r+2}}$ is bigger or equal to p then D is special. Therefore we assume from the start that $0 \leq d_i < p$, for all $i = 1, \ldots, r+2$.

The following obvious inequalities will be used throughout the rest of the paper so we record them as a separate lemma.

Lemma 1. *Let $a, b \in \mathbb{Z}$ be integers and $x \in \mathbb{R}$ any real number. Then:*

$$\frac{a}{b} - 1 < \left\lfloor \frac{a}{b} \right\rfloor \leq \frac{a}{b} \qquad \text{and} \qquad \frac{a}{b} \leq \left\lceil \frac{a}{b} \right\rceil < \frac{a}{b} + 1$$

where the equalities occur only for $a \equiv 0 \pmod{b}$, and:

$$\lfloor a + x \rfloor = a + \lfloor x \rfloor \qquad \text{and} \qquad \lceil a + x \rceil = a + \lceil x \rceil .$$

We can now state our criterion to detect when an integral degree g divisor supported on the branch locus is non-special

Theorem 2. *Let S be a compact Riemann surface and τ an automorphism of S of prime order p such that the quotient $S/\langle\tau\rangle$ has genus zero. Let $\mathrm{Fix}(\tau) = \{Q_1, \ldots, Q_{r+2}\}$ be the fixed point set of τ and let us denote by m_k the rotation number of the point Q_k.*

Then, for a divisor D of the form $D = Q_1^{d_1} \cdots Q_{r+2}^{d_{r+2}}$ with $0 \leq d_i \leq p-1$ and $\sum d_i = g$, the following four conditions are equivalent

(i) D is non-special.

(ii) $\sum_{i=1}^{r+2} \overline{d_i + m_i k} > g$, for every $k = 1, \ldots, p-1$.

(iii) $\sum_{i=1}^{r+2} \overline{d_i + m_i k} = g + p$, for every $k = 1, \ldots, p-1$.

(iv) $\sum_{i=1}^{r+2} \overline{d_i + m_i k} = g + p$, for $p-2$ integers $k \in \{1, \ldots, p-1\}$.

(v) $$\left(\sum_{i=1}^{r+1} \left\lfloor \frac{d_i + m_i k}{p} \right\rfloor \right) + \left\lfloor \frac{d_{r+2} - (\sum_{j=1}^{r+1} m_i) k}{p} \right\rfloor = -1 \ , \ \text{for all } k = 1, ..., p-1.$$

Proof. The equivalence between the first four statements was proved in our article [11]. We now see that (iii) and (v) are equivalent too.

Using the fact that for $a, b \in \mathbb{N}$ the remainder of a modulo b can be written as $\overline{a} = a - \lfloor a/b \rfloor b$, we can rewrite point (iii) as:

$$\sum d_i + k \sum m_i - p \sum \left\lfloor \frac{d_i + m_i k}{p} \right\rfloor = g + p \ , \quad \text{for every } k = 1, \ldots, p-1.$$

From the equalities $\sum d_i = g$ and $\sum m_i = np$ we deduce that

$$p \left(nk - \sum_{i=1}^{r+1} \left\lfloor \frac{d_i + m_i k}{p} \right\rfloor - \left\lfloor nk + \frac{d_{r+2} - (\sum_{j=1}^{r+1} m_i) k}{p} \right\rfloor \right) = p \, ,$$

where we have put the term $\lfloor (d_{r+2} + m_{r+2}k)/p \rfloor$ aside and written $np - \sum_{i=1}^{r+1} m_i$ instead of m_{r+2}.

Finally the result follows from the second part of Lemma 1.

$\square$

3 $\mathbb{Z}_N$-curves with no non-special divisors

For each fixed number $r \geq 2$ we will construct an infinite family of $\mathbb{Z}_N$-curves with $r+2$ ramification points, none of which will contain non-special integral divisors supported on the branch locus.

Proposition 1. *Let $r \geq 2$ be an integer and $p > 12r$ a prime. Then the $\mathbb{Z}_N$-curve of equation*

$$y^p = (x - \lambda_1) \dots (x - \lambda_r)(x - \lambda_{r+1})^{2r}(x - \lambda_{r+2})^{p-3r}$$

does not contain non-special integral divisors of degree g supported on the branch locus.

The proof will result as a consequence of two technical lemmas. We start by fixing some notation.

As above let $D = Q_1^{d_1} \cdots Q_{r+2}^{d_{r+2}}$ be an arbitrary divisor supported on the branch locus with $0 \leq d_i \leq p-1$ and $\sum d_i = g$. For $k \in \{1, \dots, p-1\}$ we introduce the following integers

$$S_k^i = S_k^i(D) = \left\lfloor \frac{d_i + k}{p} \right\rfloor, \ i = 1, \dots, r \ ,$$

$$S_k^{r+1} = S_k^{r+1}(D) = \left\lfloor \frac{d_{r+1} + 2rk}{p} \right\rfloor ,$$

$$S_k^{r+2} = S_k^{r+2}(D) = \left\lfloor \frac{d_{r+2} - 3rk}{p} \right\rfloor ,$$

We observe that for fixed $i = 1, \dots, r+1$ (resp. for $i = r+2$) S_k^i is an increasing (resp. decreasing) function of k. We also note that the distance between two consecutive values is at most 1; more precisely

$$S_k^i \leq S_{k+1}^i \leq S_{k+1}^i + 1, \qquad \text{for} \ \ i = 1, \dots, r+1;$$

and

$$S_k^{r+2} - 1 \leq S_{k+1}^{r+2} \leq S_k^{r+2}.$$

Moreover S_k^{r+2} is the only term that could possibly be negative.

Finally we denote by $S_k = S_k(D)$ the sum of all these values, that is

$$S_k = \sum_{i=1}^{r+2} S_k^i = \left(\sum_{i=1}^{r} \left\lfloor \frac{d_i + k}{p} \right\rfloor \right) + \left\lfloor \frac{d_{r+1} + 2rk}{p} \right\rfloor + \left\lfloor \frac{d_{r+2} - 3rk}{p} \right\rfloor \tag{3.1}$$

By Theorem 2 to prove Proposition 1 it is enough to see that if p is a prime such that $p \geq 12r$ then $S_k \neq -1$ for some $k \in \{1, \dots, p-1\}$.

Our next lemma detects the integers $k \in \{1, \dots, p-1\}$ for which the distance between the consecutive values S_k^i and S_{k+1}^i is exactly 1 in the two cases $i = r+1, r+2$.

Lemma 2. *Let j and ℓ be integers such that $S_1^{r+1} \leq j-1 < S_{p-1}^{r+1}$ and $S_{p-1}^{r+2} < -\ell \leq S_1^{r+2}$ and define*

$$\theta_1(j) = \left\lceil \frac{jp - d_{r+1}}{2r} \right\rceil - 1, \qquad \theta_2(\ell) = \left\lfloor \frac{p\ell + d_{r+2}}{3r} \right\rfloor.$$

Assume that $p > 3r$, then

$$S_{\theta_1(j)}^{r+1} = j - 1, \qquad S_{\theta_1(j)+1}^{r+1} = j;$$

$$S_{\theta_2(\ell)}^{r+2} = -\ell, \qquad S_{\theta_2(\ell)+1}^{r+2} = -\ell - 1.$$

Conversely, if $S_k^{r+1} = j-1$ and $S_{k+1}^{r+1} = j$ (resp. $S_k^{r+2} = -\ell$ and $S_{k+1}^{r+2} = -\ell - 1$) then $k = \theta_1(j)$ (resp. $k = \theta_2(\ell)$).

Proof. Set $k = \theta_1(j)$. Then by Lemma 1:

$$\frac{jp - d_{r+1}}{2r} - 1 \leq k < \frac{jp - d_{r+1}}{2r}$$

So we have the inequalities

$$jp - 2r \leq d_{r+1} + 2rk < jp \quad \text{and} \quad jp \leq d_{r+1} + 2r(k+1) < jp + 2r.$$

which since $p > 2r$ further implies

$$j - 1 \leq S_k^{r+1} < j \quad \text{and} \quad j - 1 \leq S_{k+1}^{r+1} < j$$

This yields $S_k^{r+1} = j - 1$ and $S_{k+1}^{r+1} = j$.

Now set $k = \theta_2(\ell)$ so that

$$\frac{p\ell + d_{r+2}}{3r} - 1 < k \leq \frac{p\ell + d_{r+2}}{3r}$$

and hence

$$-p\ell \leq d_{r+2} - 3rk < -p\ell + 3r \quad \text{and} \quad -p\ell - 3r \leq d_{r+2} - 3r(k+1) < -p\ell$$

Since $p > 3r$ this gives $S_k^{r+2} = -\ell$ and $S_{k+1}^{r+2} = -(\ell + 1)$.

The converse is a consequence of the fact that S_k^{r+1} (resp. S_k^{r+2}) is an increasing (resp. decreasing) function of k.

□

In the equation of Proposition 1, the key point in the choice of the $(r+1)$-th rotation number in relation to the r-th one is the following: if $S_k^{r+1}(D)$ and $S_k^{r+2}(D)$ increase and decrease simultaneously for the same k_1, then for the next k_2 in which $S_k^{r+1}(D)$ increases, $S_k^{r+2}(D)$ remains unchanged. This is the content of the following

Lemma 3. *With the notation as in Lemma 2, let j and ℓ be natural numbers such that $\theta_1(j) = \theta_2(\ell)$. Assume that $p > 12r$. Then there is no $h > 0$ satisfying the equality $\theta_1(j+1) = \theta_2(\ell + h)$.*

Proof. From the identities

$$k_1 := \theta_1(j) = \left\lceil \frac{jp - d_{r+1}}{2r} \right\rceil - 1 = \theta_2(\ell) = \left\lfloor \frac{p\ell + d_{r+2}}{3r} \right\rfloor.$$

we deduce that there are integers λ_1, λ_2 such that

$$\begin{array}{ll} jp - d_{r+1} = (k_1+1)\cdot 2r - \lambda_1, & 0 \le \lambda_1 \le 2r-1, \\ p\ell + d_{r+2} = k_1 \cdot 3r + \lambda_2, & 0 \le \lambda_2 \le 3r-1. \end{array} \tag{3.2}$$

Suppose that $\theta_1(j+1) = \theta_2(\ell+h)$ for some $h > 0$, then we can write

$$k_2 := \theta_1(j+1) = \left\lceil \frac{(j+1)p - d_{r+1}}{2r} \right\rceil - 1 = \theta_2(\ell+h) = \left\lfloor \frac{p(\ell+h) + d_{r+2}}{3r} \right\rfloor.$$

Now using (3.2) we get:

$$\begin{aligned} 0 &= \left\lfloor \frac{p(\ell+h) + d_{r+2}}{3r} \right\rfloor - \left\lceil \frac{(j+1)p - d_{r+1}}{2r} \right\rceil + 1 = \\ &= \left\lfloor \frac{p\ell + d_{r+2} + ph}{3r} \right\rfloor - \left\lceil \frac{jp - d_{r+1} + p}{2r} \right\rceil + 1 = \\ &= \left\lfloor k_1 + \frac{\lambda_2 + ph}{3r} \right\rfloor - \left\lceil k_1 + 1 + \frac{p - \lambda_1}{2r} \right\rceil + 1 = \\ &= \left\lfloor \frac{\lambda_2 + ph}{3r} \right\rfloor - \left\lceil \frac{p - \lambda_1}{2r} \right\rceil = \\ &= \left\lfloor \frac{\lambda_2 + ph}{3r} \right\rfloor - \left\lfloor \frac{p}{2r} \right\rfloor - \left\lceil \frac{\widetilde{p} - \lambda_1}{2r} \right\rceil, \end{aligned}$$

where $\widetilde{p}$ is the remainder of p modulo $2r$ and so $p = \left\lfloor \frac{p}{2r} \right\rfloor 2r + \widetilde{p}$, with $0 \le \widetilde{p} \le 2r-1$. It follows that

$$\left\lfloor \frac{\lambda_2 + ph}{3r} \right\rfloor - \left\lfloor \frac{p}{2r} \right\rfloor = \left\lceil \frac{\widetilde{p} - \lambda_1}{2r} \right\rceil, \qquad \text{for some } h.$$

In order to show that this last equality leads to a contradiction we note that $\left\lceil \frac{\widetilde{p} - \lambda_1}{2r} \right\rceil$ can only be 0 or 1.

Now if $h = 1$ then by Lemma 1 and using the condition on p:

$$\begin{aligned} \left\lfloor \frac{\lambda_2 + p}{3r} \right\rfloor - \left\lfloor \frac{p}{2r} \right\rfloor &< \frac{\lambda_2 + p}{3r} - \frac{p}{2r} + 1 = \frac{2\lambda_2 + 2p - 3p}{6r} + 1 < \\ &< \frac{-p}{6r} + 2 < 0. \end{aligned}$$

On the other hand if $h \ge 2$:

$$\begin{aligned} \left\lfloor \frac{\lambda_2 + hp}{3r} \right\rfloor - \left\lfloor \frac{p}{2r} \right\rfloor &> \left\lfloor \frac{\lambda_2 + 2p}{3r} \right\rfloor - \left\lfloor \frac{p}{2r} \right\rfloor > \\ &> \frac{\lambda_2 + 2p}{3r} - 1 - \frac{p}{2r} = \frac{2\lambda_2 + 4p - 3p}{6r} - 1 > \\ &> \frac{p}{6r} - 1 > 1. \end{aligned}$$

So, in any case we get a contradiction.

□

We are now in position to provide the

of Proposition 1. We have to show that for some $k = 1, \dots, p-1$ the sum $S_k = S_k(D)$ defined in (3.1) is different from -1.

Since by definition $S_k^i \geq 0$ for $i \leq r+1$ and, by our hypothesis on p, $S_k^{r+2} \geq -1$, we can assume that $S_1^{r+2}(D) = -1$ and that $S_1^i(D) = 0$, for $i = 1, \dots, r+1$, since otherwise the sum $S_k(D)$ would be different from -1 already for $k = 1$.

In particular $S_1^{r+1} = 0$, and since $r \geq 2$ we have:

$$S_{p-1}^{r+1} = \left\lfloor \frac{2r(p-1) + d_{r+1}}{p} \right\rfloor \geq \left\lfloor \frac{4p - 4 + d_{r+1}}{p} \right\rfloor > 1$$

Hence we can define the indexes $k_1 := \theta_1(1)$ and $k_2 := \theta_1(2)$ as in Lemma 2. For the first one we have that $S_{k_1}^{r+1}(D) = 0$ and $S_{k_1+1}^{r+1}(D) = 1$. As we are assuming that D is non-special the following identity holds

$$S_{k_1}(D) = \sum_{i=1}^{r} S_{k_1}^i(D) + S_{k_1}^{r+1}(D) + S_{k_1}^{r+2}(D) = \xi_1 + S_{k_1}^{r+2}(D) = -1,$$

where $\xi_1 = \sum_{i=1}^{r} S_{k_1}^i(D)$, and so $S_{k_1}^{r+2}(D) = -(\xi_1 + 1)$.

But then for $k_1 + 1$ we have:

$$S_{k_1+1}(D) = \xi_1' + 1 + S_{k_1+1}^{r+2}(D) = -1,$$

hence $S_{k_1+1}^{r+2}(D) = -(\xi_1' + 2)$. Now, since $\xi_1' = \sum_{i=1}^{r} S_{k_1+1}^i(D) \geq \xi_1$ and the integers $S_{k_1}^{r+2}$, $S_{k_1+1}^{r+2}$ differ by at most a unity it follows that $\xi_1' = \xi_1$ and $S_{k_1+1}^{r+2}(D) = S_{k_1}^{r+2}(D) - 1$. By Lemma 2 the index k_1 must be of the form $k_1 = \theta_2(\ell)$ for some ℓ (in fact for $\ell = \xi_1 + 1$).

As for k_2 we have $S_{k_2}^{r+1}(D) = 1$ and $S_{k_2+1}^{r+1}(D) = 2$ and so:

$$S_{k_2}(D) = \sum_{i=1}^{r} S_{k_2}^i(D) + S_{k_2}^{r+1}(D) + S_{k_2}^{r+2}(D) = \xi_2 + 1 + S_{k_2}^{r+2}(D) = -1,$$

where $\xi_2 = \sum_{i=1}^{r} S_{k_2}^i(D)$. Then $S_{k_2}^{r+2}(D) = -(\xi_2 + 2)$. We have as well:

$$S_{k_2+1}(D) = \xi_2' + 2 + S_{k_2+1}^{r+2}(D) = -1,$$

so $S_{k_2+1}^{r+2}(D) = -(\xi_2' + 3)$. As above we conclude that $\xi_2' = \xi_2$, hence k_2 must be of the form $k_2 = \theta_2(\ell + h)$ for some $h > 0$.

So $k_1 = \theta_1(1) = \theta_2(\ell)$ and $k_2 = \theta_1(2) = \theta_2(\ell + h)$. But this yields a contradiction with Lemma 3.

□

In fact with little changes in the previous proofs, we can find a stronger result, which in addition is also valid for curves with only 3 ramification values:

Proposition 2. *Let $r \geq 1$ be an integer and p a prime. Let S be the $\mathbb{Z}_N$-curve of equation*

$$y^p = (x-\lambda_1)^{m_1} \dots (x-\lambda_r)^{m_r} (x-\lambda_{r+1})^m (x-\lambda_{r+2})^{p-m-M},$$

where set $M = \sum_{i=1}^r m_i$ and $m = m_{r+1}$.
Suppose that the following conditions are satisfied:

(i) $m > M$ and $m \geq 3$;

(ii) $p > \max\left\{ \frac{2m(m+M)}{M}, \frac{2m(m+M)}{m-M} \right\}$.

Then S does not contain non-special integral divisors of degree g supported on the branch locus.

Proof. The proof of the previous proposition can be mimicked until the last part. In fact the conditions on m ensure that

$$S_{p-1}^{r+1} \equiv \left\lfloor \frac{m(p-1)+d_{r+1}}{p} \right\rfloor \geq \left\lfloor \frac{3p-3+d_{r+1}}{p} \right\rfloor > 1$$

so the indexes k_1 and k_2 can be defined as in Proposition 1.

Finally one gets the following equality (similar to the one in Lemma 3):

$$\left\lfloor \frac{\lambda_2 + ph}{m+M} \right\rfloor - \left\lfloor \frac{p}{m} \right\rfloor = \left\lceil \frac{\widetilde{p} - \lambda_1}{m} \right\rceil, \qquad \text{for some } h,$$

where $0 \leq \widetilde{p}, \lambda_1 \leq m-1$ and $0 \leq \lambda_2 \leq m+M-1$. Once again the right hand of the equality is either 0 or 1, and for $h=1$ we get:

$$\begin{aligned} \left\lfloor \frac{\lambda_2 + p}{m+M} \right\rfloor - \left\lfloor \frac{p}{m} \right\rfloor &< \frac{\lambda_2 + p}{m+M} - \frac{p}{m} + 1 = \frac{m\lambda_2 + mp - (m+M)p}{m(m+M)} + 1 < \\ &< \frac{-Mp}{m(m+M)} + 2 < 0, \qquad \text{if } p > \frac{2m(m+M)}{M}, \end{aligned}$$

whereas for $h=2$:

$$\begin{aligned} \left\lfloor \frac{\lambda_2 + 2p}{m+M} \right\rfloor - \left\lfloor \frac{p}{m} \right\rfloor &> \frac{\lambda_2 + 2p}{m+M} - 1 - \frac{p}{m} = \frac{m\lambda_2 + 2mp - (m+M)p}{m(m+M)} - 1 > \\ &> \frac{m-M}{m(m+M)} p - 1 > 1, \qquad \text{if } p > \frac{2m(m+M)}{m-M}. \end{aligned}$$

Taking p bigger than the maximum of the two bounds, we get the result. Note that we need the $m - M > 0$ condition in the last inequality.

$\square$

4 Non-special divisors of $\mathbb{Z}_N$-curves for some small values of N and s

In this section we give complete lists of all non-special integral divisors supported on the branch locus for some $\mathbb{Z}_N$-curves with small values of the degree N and of the number of branching values s.

The list has been obtained by checking Theorem 2 by computer means.

Remember that the branch locus of a $\mathbb{Z}_N$-curve of the form

$$y^p = (x-\lambda_1)^{m_1} \dots (x-\lambda_s)^{m_s}$$

is the set $\{Q_i = (\lambda_i, 0) \ : \ i = 1, \dots, s\}$.

Example 1. *(Trigonal curves with $s = 4$)*
Any trigonal curve with 4 ramification values can be written as

$$y^3 = (x-\lambda_1)(x-\lambda_2)(x-\lambda_3)^2(x-\lambda_4)^2$$

The list of all non-special integral divisors supported on the branch locus is

$$D \ = \ \{ \ Q_i^2, \ Q_j^2, \ Q_iQ_j \ : \ i = 1, 2; \ j = 3, 4 \ \}$$

Example 2. *(Trigonal curves with $s = 5$)*
There is only one kind of trigonal curve with 5 ramification values

$$y^3 = (x-\lambda_1)(x-\lambda_2)(x-\lambda_3)(x-\lambda_4)(x-\lambda_5)^2$$

The list of all non-special integral divisors supported on the branch locus is

$$D \ = \ \{ \ Q_iQ_j^2, \ Q_iQ_5^2, \ Q_iQ_jQ_5 \ : \ i, j = 1, 2, 3, 4; \ i \neq j \ \}$$

Example 3. *($\mathbb{Z}_N$-curves for $N = 5$ and $s = 4$)*
There are three different kinds of $\mathbb{Z}_N$-curves for $N = 5$ with 4 ramification values

- $y^5 = (x-\lambda_1)(x-\lambda_2)(x-\lambda_3)(x-\lambda_4)^2$
  The list of all non-special integral divisors supported on the branch locus is:
  $$D \ = \ \{ \ Q_i^2Q_4^2, \ Q_iQ_j^3, \ Q_i^2Q_jQ_4 \ : \ i, j = 1, 2, 3; \ i \neq j \ \}$$
- $y^5 = (x-\lambda_1)(x-\lambda_2)(x-\lambda_3)^4(x-\lambda_4)^4$
  The list of all non-special integral divisors supported on the branch locus is:
  $$D \ = \ \{ \ Q_i^4, \ Q_j^4, \ Q_i^3Q_j, \ Q_iQ_j^3, \ Q_i^2Q_j^2 \ : \ i = 1, 2; \ j = 3, 4 \ \}$$
- $y^5 = (x-\lambda_1)(x-\lambda_2)^2(x-\lambda_3)^3(x-\lambda_4)^4$
  The list of all non-special integral divisors supported on the branch locus is:
  $$\begin{aligned} D \ = \ \{ \ & Q_1^4, \ Q_2^4, \ Q_3^4, \ Q_4^4, \ Q_1Q_4^3, \ Q_1^3Q_4, \ Q_2Q_3^3, \ Q_2^3Q_3, \ Q_1^2Q_4^2, \ Q_2^2Q_3^2, \\ & Q_1Q_2Q_3^2, \ Q_1Q_3Q_4^2, \ Q_1^2Q_2Q_4, \ Q_2^2Q_3Q_4, \ Q_1Q_2Q_3Q_4 \ \} \end{aligned}$$

Example 4. *($\mathbb{Z}_N$-curves for $N = 5$ and $s = 5$)*
There are three different kinds of $\mathbb{Z}_N$-curves for $N = 5$ with 5 ramification values

- $y^5 = (x-\lambda_1)(x-\lambda_2)(x-\lambda_3)(x-\lambda_4)(x-\lambda_5)$
  The list of all non-special integral divisors supported on the branch locus is:

$$D \;=\; \{\; Q_iQ_j^2Q_k^3 \;:\; i,j,k = 1,2,3,4,5;\; i \neq j,\; i \neq k,\; j \neq k \;\}$$

- $y^5 = (x-\lambda_1)(x-\lambda_2)(x-\lambda_3)(x-\lambda_4)^3(x-\lambda_5)^4$
  The list of all non-special integral divisors supported on the branch locus is:

$$\begin{aligned} D \;=\; \{\; & Q_i^2Q_j^4,\; Q_i^2Q_4^4,\; Q_i^2Q_5^4,\; Q_iQ_jQ_4Q_5^3,\; Q_iQ_j^2Q_4Q_5^2,\; Q_iQ_j^2Q_5^3, \\ & Q_iQ_j^3Q_4,\; Q_iQ_j^3Q_4^2,\; Q_iQ_j^4Q_4,\; Q_iQ_4Q_5^4,\; Q_i^2Q_j^2Q_5^2,\; Q_i^2Q_j^3Q_5, \\ & Q_i^3Q_4^3 \;:\; i,j = 1,2,3;\; i \neq j \;\} \end{aligned}$$

- $y^5 = (x-\lambda_1)(x-\lambda_2)(x-\lambda_3)^2(x-\lambda_4)^2(x-\lambda_5)^4$
  The list of all non-special integral divisors supported on the branch locus is:

$$\begin{aligned} D \;=\; \{\; & Q_{j_1}^2Q_5^4,\; Q_{j_1}^2Q_{i_1}^4,\; Q_{j_1}^2Q_{j_2}^4,\; Q_{i_1}^3Q_{j_1}^3,\; Q_{i_1}Q_{j_1}Q_5^4,\; Q_{i_1}Q_{j_1}^2Q_5^3, \\ & Q_{i_1}^2Q_{j_1}^2Q_5^2,\; Q_{i_1}^3Q_{j_1}^3Q_5,\; Q_{i_1}Q_{j_1}Q_{j_2}^4,\; Q_{i_1}^2Q_{j_1}Q_{j_2}^3,\; Q_{i_1}Q_{i_2}^3Q_5^2, \\ & Q_{i_1}^2Q_{j_1}Q_{j_2}^2Q_5,\; Q_{i_1}Q_{i_2}Q_{j_1}Q_5^3,\; Q_{i_1}Q_{i_2}^2Q_{j_1}Q_5^2,\; Q_{i_1}Q_{i_2}^3Q_{j_1}Q_5, \\ & Q_{i_1}Q_{i_2}^2Q_{j_1}Q_{j_2}Q_5 \;:\; i_1,i_2 = 1,2;\; j_1,j_2 = 3,4;\; i_1 \neq i_2,\; j_1 \neq j_2 \;\} \end{aligned}$$

We observe that the condition on p in Proposition 1 is essential. Next we give the list of non-special integral divisors supported on their branch locus of some curves of the same type as those considered there, namely

$$y^p = (x-\lambda_1)\ldots(x-\lambda_r)(x-\lambda_{r+1})^{2r}(x-\lambda_{r+2})^{p-3r}, \quad 3r < p \leq 12r,$$

- For $r = 2$:
  1. $y^7 = (x-a_1)(x-a_2)(x-a_3)^4(x-a_4)^1$

$$\begin{aligned} D \;=\; \{\; & Q_iQ_j^3Q_3^2,\; Q_iQ_j^3Q_3^2,\; Q_iQ_j^4Q_3,\; Q_i^2Q_j^4 \;: \\ & i,j = 1,2,4;\; i \neq j \;\} \end{aligned}$$

  2. $y^{11} = (x-a_1)(x-a_2)(x-a_3)^4(x-a_4)^5$

$$\begin{aligned} D \;=\; \{\; & Q_iQ_j^4Q_3^2Q_4^3,\; Q_iQ_j^5Q_3^2Q_4^2,\; Q_iQ_j^6Q_3^1Q_4^2,\; Q_iQ_j^7Q_3Q_4 \;: \\ & i,j = 1,2;\; i \neq j \;\} \end{aligned}$$

  3. $y^{13} = (x-a_1)(x-a_2)(x-a_3)^4(x-a_4)^7$

$$D \;=\; \{\; Q_1^1Q_2^5Q_3^2Q_4^4,\; Q_1^1Q_2^8Q_3^1Q_4^2,\; Q_1^5Q_2^1Q_3^2Q_4^4,\; Q_1^8Q_2^1Q_3^1Q_4^2 \;\}$$

- For $r = 3$:

1. $y^{11} = (x-a_1)(x-a_2)(x-a_3)(x-a_4)^6(x-a_5)^2$

$$\begin{aligned} D \;=\; \{ & Q_iQ_j^3Q_k^8Q_4^3,\ Q_iQ_j^3Q_k^8Q_4^2Q_5,\ Q_iQ_j^3Q_k^7Q_4^4,\ Q_i^2Q_j^4Q_k^8Q_5, \\ & Q_iQ_j^4Q_k^8Q_4Q_5,\ Q_iQ_j^5Q_4^3Q_5^6,\ Q_iQ_j^6Q_4^2Q_5^6,\ Q_i^3Q_j^5Q_4^4Q_5^3, \\ & Q_i^3Q_j^8Q_4^2Q_5^2,\ Q_i^4Q_j^8Q_4Q_5^2,\ Q_i^4Q_j^6Q_5^5 \ :\ i,j,k=1,2,3; \\ & i\neq j,\ i\neq k,\ j\neq k \ \} \end{aligned}$$

2. $y^{13} = (x-a_1)(x-a_2)(x-a_3)(x-a_4)^6(x-a_5)^4$

$$\begin{aligned} D \;=\; \{ & Q_iQ_j^4Q_k^8Q_4^4Q_5,\ Q_iQ_j^5Q_k^8Q_2^2Q_5^2,\ Q_iQ_j^5Q_k^8Q_4^3Q_5, \\ & Q_iQ_j^5Q_k^9Q_4Q_5^2 \ :\ i,j,k=1,2,3;\ i\neq j,\ i\neq k,\ j\neq k \ \} \end{aligned}$$

Acknowledgement. *We would like to thank Hershel M. Farkas for the fruitful exchange of correspondence on this matter, and in particular for the observation that the curve $y^{17} = (x-\lambda_1)(x-\lambda_2)^2(x-\lambda_3)^{14}$ has no non-special integral divisors supported on the branch locus, a fact that led us to consider the problem studied in this paper. We must also thank Adrián Ubis for his priceless hints on how to handle the discrete calculations found throughout the text.*

The first author was partially supported by MEC grant MTM2006-01859. The second author was partially supported by an FPU grant of the MICINN.

References

[1] Belavin, A. A. and Knizhnik, V. G.: Complex geometry and the theory of quantum strings, Zh. Eksp. Teor. Fiz. **91**, 364–390 (1986).

[2] Bershadsky, M. and Radul, A.: Conformal field theories with additional Z_N symmetry, Int. J. Mod. Phys. **2**, 165–178 (1987).

[3] Bershadsky, M. and Radul, A.: Fermionic fields on Z_N-curves. Comm. Math. Phys. **116**, no. 4, 689-700 (1988).

[4] Ebin, D. G. and Farkas, H. M.: Thomae's formulae for Z_N-curves, to appear in Journal d'Analyse Mathmatique.

[5] Eisenmann, A. and Farkas, H. M.: An elementary proof of Thomae's formulae, Online Journal of Analytic Combinatorics, No. 3 (2008).

[6] Enolski, V. Z. and Grava, T.: Thomae type formulae for singular Z_N curves, Lett. Math. Phys. **76**, no. 2-3, 187-214 (2006).

[7] Farkas, H. M. and Kra, I.: Riemann surfaces. Graduate Texts in Mathematics, **71**. Springer-Verlag, New York-Berlin, 1980.

[8] Farkas, H. M. and Zemel, S.: Generalizations of Thomae's formula for Z_n curves. Developments in Mathematics, **21**. Springer, New York, 2011.

[9] Fay, John D.: Theta functions on Riemann surfaces. Lecture Notes in Mathematics, Vol. 352. Springer-Verlag, Berlin-New York 1973.

[10] González-Díez, G.: Loci of curves which are prime Galois coverings of $\mathbb{P}^1$, Proc. London Math. Soc. (3) 62, no. **3**, 469-489 (1991).

[11] González-Díez, G.: Non-special divisors supported on the branch set of a $p-$gonal Riemann surface, in Geometry of Riemann surfaces, London Math. Soc., Lecture Note Ser. 368, 248-269 (2010).

[12] Kopeliovich, Y.: Thomae formula for general cyclic covers of $\mathbb{CP}^1$, Lett. Math. Phys. **94**, no. 3, 313-333 (2010).

[13] Mumford, D.: Tata lectures on theta I and II, Birkhauser, New York 1983.

[14] Nakayashiki, A.: On the Thomae formula for Z_N curves, Publ. Res. Inst. Math. Sci. **33**, **6**, 987-1015 (1997).

[15] Thomae, J.: Bestimmung von dlg $\vartheta(0,\ldots,0)$ durch die Klassenmoduln, Crelle's Journal, **66**, 92-96 (1866).

[16] Thomae, J.: Beitrag zur bestimug von $\theta(0,\ldots,0)$ durch die Klassenmoduln algebraischer Funktionen, Crelle's Journal, **71** (1870).