Goethe-Universität Frankfurt
Institut für Mathematik
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Elementare Zahlentheorie
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Übungsblatt 4

Aufgabe 1 (4 Punkte)

Let $\mathbb{C}[[T]]$ be the ring of formal power series.

(a) Show that $\mathbb{C}[[T]]^\times = \{\sum_{a=0}^{\infty} a_i T^i \mid a_0 \in \mathbb{C}^\times\}$.

(b) Calculate $(1-T)^{-1}$.

(c) Let $f\colon \mathbb{N} \to \mathbb{C}$ be a multiplicative function. We say that f is invertible w.r.t. convolution (Faltung) if there exists a $g\colon \mathbb{N} \to \mathbb{C}$ such that $f * g = \delta$.

Show that f is invertible w.r.t. convolution if and only if $f \neq 0$ and that the inverse of f is multiplicative.

Aufgabe 2 (4 Punkte)

Show the following identities of arithmetic functions:

(a) $\sigma(n) = \prod_{p|n} \frac{p^{\nu_p(n)+1}-1}{p-1}$;

(b) $\sum_{d|n} \tau(d)^3 = \left(\sum_{d|n} \tau(d)\right)^2$.

Aufgabe 3 (2 Punkte)

(a) Let $f\colon \mathbb{N} \to \mathbb{C}$ be a multiplicative function. Show that, for all $n \in \mathbb{N}$

$$\sum_{d|n} \mu(d) f(d) = \prod_{p|n} (1 - f(p)).$$

(b) Show that this implies $\frac{\varphi(n)}{n} = \prod_{p|n}\left(1 - \frac{1}{p}\right)$ for all $n \in \mathbb{N}$.

Aufgabe 4 (6 Punkte)

We say that a poset $(\mathcal{P}, \leq)$ is *locally finite* if $[a, b] := \{x : a \leq x \leq b\}$ is finite for every $a, b \in \mathcal{P}$. For a locally finite poset $(\mathcal{P}, \leq)$, we denote the set of intervals $\mathrm{Int}(\mathcal{P})$ and for any commutative ring R, we define the *incidence algebra* of $(\mathcal{P}, \leq)$ to be the set of maps $f\colon \mathrm{Int}(\mathcal{P}) \to R$ with multiplication

$$f * g([a, b]) := \sum_{x \in [a,b]} f([a, x])g([x, b]).$$

The multiplicative identity element is the δ function where $\delta([a, b]) = 1$ if $a = b$ and 0 otherwise. As usual, we define $\mathbb{1}$ to be the constant function 1.

(a) Let μ be the element of the incidence algebra given by

$$\mu([a, b]) = \begin{cases} 1, & \text{if } a = b, \\ -\sum_{a \leq x < b} \mu([a, x]), & \text{if } a < b. \end{cases}$$

Show that $\mu * \mathbb{1} = \delta$.

(b) Let $(\mathcal{P}, \leq)$ be the set $\mathbb{N}$ with $a \leq b :\iff a \mid b$. Show that a subalgebra of the incidence algebra of functions on the intervals $[1, a]$, $a \in \mathbb{N}$, corresponds to arithmetic functions with the Dirichlet involution and that μ corresponds to the Möbius function.

(c) Let X be a set and $(\mathcal{P}, \leq)$ the finite subsets of X ordered by inclusion.

(i) Show that $\mu([A, B]) = (-1)^{|B \setminus A|}$ for $A \subseteq B$.

Hint: Induction and binomial theorem.

(ii) Show the inclusion-exclusion formula using Möbius inversion.

That is, let $A_1, \ldots, A_n \subseteq X$ and $I \subseteq \{1, \ldots, n\}$. We define

$$A_I := \bigcap_{i \in I} A_i \quad \text{with} \quad A_\emptyset = X.$$

Then $|X \setminus (A_1 \cup \cdots \cup A_n)| = \sum_{I \subseteq \{1,\ldots,n\}} (-1)^{|I|} |A_I|$.

Hint: Consider $f(I) = |A_I \setminus \bigcup_{I \subset J} A_J|$.

Please, upload your solutions on the Olat page of this course, by **14:00** on **Wednesday, 20.5.2020**.