**Goethe-Universität Frankfurt**
**Institut für Mathematik**
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**Elementare Zahlentheorie**
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# Übungsblatt 5

## Aufgabe 1 (4 Punkte)

(i) Compute the continued fraction expansions of $\frac{15}{11}$, $\varphi^{-1}$, $\sqrt{10}$ and $\sqrt{3}$ via the Kettenbruchalgorithmus, where $\varphi$ is the golden ratio.

(ii) Determine the values of the continued fractions $[2, 3, 5, 2]$ and $[2, 1, 4, 2, 1, 4, \dots]$.

## Aufgabe 2 (2 Punkte)

We denote by $L_n$ the $n$-th Fibonacci number; see Homework 1 for the definition.

(i) Compute the continued fraction expansion of $\frac{L_{n+1}}{L_n}$ for every $n \in \mathbb{Z}_{\geq 0}$.

(ii) Compute $\lim_{n\to\infty} \frac{L_{n+1}}{L_n}$.

## Aufgabe 3 (4 Punkte)

Let $(a_i)_{i\in\mathbb{Z}_{\geq 0}}$ be a sequence of integers such that $a_i \geq 1$ for every $i \geq 1$. For every $n \in \mathbb{Z}_{\geq 0}$, we define as in the lesson

$$M_n = \begin{pmatrix} p_n & r_n \\ q_n & s_n \end{pmatrix} := \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix},$$

such that the *$n$-th convergent* of $[a_0, a_1, \dots]$ is $\frac{p_n}{q_n} = [a_0, a_1, ..., a_n]$. Prove that:

(i) $[a_n, a_{n-1}, \dots, a_1, a_0] = \frac{p_n}{p_{n-1}}$ for every $n \in \mathbb{Z}_{\geq 0}$.

(ii) $[a_n, a_{n-1}, \dots, a_1] = \frac{q_n}{q_{n-1}}$ for every $n \in \mathbb{Z}_{\geq 0}$.

## Aufgabe 4 (6 Punkte)

The goal of this exercise is to prove that the continued fraction expansion of $e$ is

$$[1, 0, 1, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, \dots]. \tag{1}$$

For every $n \in \mathbb{Z}_{\geq 0}$, we define

$$A_n := \int_0^1 \frac{x^n(x-1)^n}{n!} e^x dx, \; B_n := \int_0^1 \frac{x^{n+1}(x-1)^n}{n!} e^x dx, \; C_n := \int_0^1 \frac{x^n(x-1)^{n+1}}{n!} e^x dx.$$

Let $p_n$ and $q_n$ be as in Aufgabe 3, such that $\frac{p_n}{q_n}$ is the *$n$-th convergent* of the continued fraction (1). (Notice that $q_1 = 0$, so $p_1/q_1$ is undefined, but it will not be a problem.)

(i) Prove the following recurrence relations:

$$\begin{aligned} p_{3n} &= p_{3n-1} + p_{3n-2}, & q_{3n} &= q_{3n-1} + q_{3n-2}, \\ p_{3n+1} &= 2np_{3n} + p_{3n-1}, & q_{3n+1} &= 2nq_{3n} + q_{3n-1}, \\ p_{3n+2} &= p_{3n+1} + p_{3n}, & q_{3n+2} &= q_{3n+1} + q_{3n}. \end{aligned}$$

(ii) Prove that $A_n = -B_{n-1} - C_{n-1}$, $B_n = -2nA_n + C_{n-1}$ and $C_n = B_n - A_n$.
**Hint:** compute the derivative of $x^n(x-1)^n e^x/n!$ and $x^n(x-1)^{n+1}e^x/n!$.

(iii) Prove that $A_n = q_{3n}e - p_{3n}$, $B_n = p_{3n+1} - q_{3n+1}e$, and $C_n = p_{3n+2} - q_{3n+2}e$.

(iv) Prove that $\lim_{n\to\infty} \frac{p_n}{q_n} = e$.

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Please, upload your solutions on the Olat page of this course, by **14:00** on **Wednesday, 27.5.2020**.