Goethe-Universität Frankfurt
Institut für Mathematik
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Elementare Zahlentheorie
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Übungsblatt 8

Aufgabe 1 (2 Punkte)

Let $f = ax^2 + bxy + cy^2$ with $D = b^2 - 4ac > 0$ be an indefinite quadratic form (i.e. $D > 0$).

We call f *reduced* if $a > 0$, $c > 0$, and $b > a + c$.

Show that if $D > 0$ is not a square there are only finitely many reduced forms with discriminant D.

Aufgabe 2 (8 Punkte)

Let $f = ax^2 + bxy + cy^2 =: [a, b, c]$ with $D = b^2 - 4ac > 0$ be an indefinite quadratic form and $S_n = \begin{pmatrix} n & 1 \\ -1 & 0 \end{pmatrix}$.

For an indefinite quadratic form f, we define $T(f) \coloneqq f|_{S_n}$ for $n = \lfloor \frac{b+\sqrt{D}}{2a} \rfloor + 1$.

(a) Show that $T(f)$ is an indefinite quadratic form.

(b) Show that $[-1, 6, -3]$ is not in the T-orbit of $[1, 6, 3]$.

(c) Show that, for any indefinite form f, the T-orbit of f contains a reduced form.

(d) Show that $T(f)$ is reduced if f is reduced.

(e) Show that any two equivalent reduced forms are in the same T-orbit.

Hint: For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, express the coefficients of $f|_A$ in terms of f and do a case distinction for the sign of c.

Aufgabe 3 (6 Punkte)

For a series of integers $n_0, n_1, n_2, \ldots$ with $n_i \geq 2$ for all $i \geq 1$, we define

$$[[n_0, n_1, \ldots, n_s]] \coloneqq n_0 - \cfrac{1}{n_1 - \cfrac{1}{n_2 - \ddots - \frac{1}{n_s}}} \quad \text{and} \quad [[n_0, n_1, \ldots]] \coloneqq \lim_{s \to \infty} [[n_0, n_1, \ldots, n_s]]$$

Setting, for $w \in \mathbb{R}$, $n_0 = \lfloor w \rfloor + 1$, $w_1 = \frac{1}{n_0 - w}$ and inductively $n_i = \lfloor w_i \rfloor + 1$ and $w_{i+1} = \frac{1}{n_i - w_i}$ gives a bijection to the real numbers via $w = [[n_0, n_1, n_2, \ldots]]$.

(a) Compute $[[2, 2, \ldots]]$.

(b) Show that $w \in \mathbb{Q} \iff$ there exists a j such that $n_i = 2$ for all $i > j$.

(c) Show that there exists $Q \in \mathbb{Z}[X]$ of degree 2 with $Q(w) = 0 \iff$ there exists $r \geq 1$ and j such that $n_{i+r} = n_i$ for all $i > j$.

Hint: For an indefinite quadratic form f, relate the expansion of the roots of $f(x, -1)$ with the roots of $T(f)(x, -1)$ and use Aufgabe 2.

(d) Show that there exists an $r \geq 1$ and $n_{i+r} = n_i$ for all $i \iff w$ is the larger zero of $ax^2 + bx + c$, where $[a, b, c]$ is a reduced indefinite quadratic form.

Please, upload your solutions on the Olat page of this course, by **14:00** on **Wednesday, 17.06.2020**.