Goethe-Universität Frankfurt
Institut für Mathematik
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Elementare Zahlentheorie
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Übungsblatt 9

Aufgabe 1 (4 Punkte)

We denote by $[[n_0, n_1, n_2, \dots]]$ the *negative* continued fractions, as in Übungsblatt 8 Aufgabe 3.
Let $(a_j)_{j\geq 0}$ be a sequence of integers such that $a_1, a_2, \dots \geq 1$. Prove that

$$[a_0, a_1, a_2, \dots] = [[a_0+1, \underbrace{2, \dots, 2}_{a_1-1}, a_2+2, \underbrace{2, \dots, 2}_{a_3-1}, a_4+2, \dots]].$$

Aufgabe 2 (6 Punkte)

Let d be a positive integer that is not a square. We denote by p_n/q_n the n-th convergent of the continued fraction expansion of $\sqrt{d}$, and by s the length of the period of that continued fraction. The goal of this guided exercise is to prove that the *positive* solutions (i.e. such that *both* entries are positive) of Pell's equation

$$X^2 - dY^2 = 1 \tag{1}$$

are the pairs (p_{ns-1}, q_{ns-1}) if s is even, and (p_{2ns-1}, q_{2ns-1}) if s is odd, where $n \geq 1$.

(i) Prove that the pairs appearing above are positive solutions of (1).

(ii) We want to prove the converse. Let (t, u) be a positive solution of (1), and let $[b_0, \dots, b_{k-1}]$ be the continued fraction expansion of t/u. Check that we can suppose $k > 0$ to be even.

(iii) Let t'/u' be the last but one convergent of $[b_0, \dots, b_{k-1}]$. Show that

$$t(u' - t) = u(t' - du).$$

(iv) Show that $u' = t - b_0 u$ and $t' = du - b_0 t$.

(v) Deduce that $\frac{t(b_0+\sqrt{d})+t'}{u(b_0+\sqrt{d})+u'} = \sqrt{d}$. Conclude computing the continued fraction expansion of the left-hand side.

Aufgabe 3 (2 Punkte)

Let d be a positive integer that is not a square. We denote by p_n/q_n the n-th convergent of the continued fraction expansion of $\sqrt{d}$, and by s the length of the period of that continued fraction.
Suppose that s is odd. Prove that the *positive* solutions of Pell's equation

$$X^2 - dY^2 = -1$$

are the pairs $(p_{(2n-1)s-1}, q_{(2n-1)s-1})$ for $n \geq 1$. Show that there are no positive solutions if s is even.
Hint. Modify the argument of the previous exercise, supposing $k > 0$ odd. Notice that the case $k = 1$ must be treated separately.

Aufgabe 4 (4 Punkte)

Let d be a positive integer that is not a square, and let (α, β) be the *minimal solution* of Pell's equation $X^2 - dY^2 = 1$, that is, the positive solution with first entry α minimal. Prove that if Pell's equation $X^2 - dY^2 = -1$ has a solution, then its minimal solution (a, b) satisfies:

$$\alpha = 2a^2 + 1 \qquad \text{and} \qquad \beta = 2ab.$$

Hint. Let $\zeta = \alpha + \beta\sqrt{d}$ and $z = a + b\sqrt{d}$. Prove that $1 < z < \zeta$ to deduce that $z^2 = \zeta$.

Please, upload your solutions on the Olat page of this course, by **14:00** on **Wednesday, 24.06.2020**.