The moduli space of multi-scale differentials

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(joint with M. Costantini and J. Zachhuber)
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Overview:

- Properties of the moduli space of multi-scale differentials (BCGGM, 2019)

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- Equivalence classes of prong-matchings. (BCGGM, 2019)
- Tautological ring (CMZ, 2020)
- Chern classes and Euler characteristics. (CMZ, 2020)

Let

$$\mathcal{M}_{g,n} = \{(X, p_1, \dots, p_n) \colon p_i \in X\}$$

be the moduli space of n-pointed Riemann surfaces of genus g.
Let

$$\Omega\mathcal{M}_{g,n}(\mu) = \left\{(X, \omega, p_1, \dots, p_n) \colon \operatorname{div}(\omega) = \sum_{i=1}^{n} m_i p_i\right\}$$

be the stratum of meromorphic abelian differentials of profile $\mu = (m_1, \dots, m_n)$ with $m_i \in \mathbb{Z}$ and $\sum_{i=1}^{n} m_i = 2g - 2$.

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be the **projective** stratum of meromorphic abelian differentials of profile $\mu = (m_1, \ldots, m_n)$ with $m_i \in \mathbb{Z}$ and $\sum_{i=1}^{n} m_i = 2g - 2$.

Let $\overline{\mathcal{M}}_{g,n}$ be the Deligne-Mumford compactification of $\mathcal{M}_{g,n}$. It is the moduli space of n-pointed stable curves of genus g.
It has very nice properties:

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- It is an orbifold/**smooth proper** Deligne-Mumford stack.
- The boundary is a **normal crossing divisor**.
- The boundary strata are indexed by **stable graphs**, i.e. the dual graphs of stable curves of genus g.
  The **codimension** of a boundary component is given by the **number of edges** of the graph.
  Each boundary stratum is almost a **product of moduli spaces of curves of lower complexity**.

Theorem (BCGGM, 2019) For each stratum $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ of meromorphic abelian differentials there exists a compactification $\mathbb{P}\Xi\mathcal{M}_{g,n}(\mu)$, the moduli space of multi-scale differentials, with the following properties:

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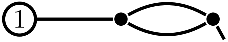
Examples of boundary strata in $\overline{\mathcal{M}}_{2,1}$

$$\overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{1,2}$$

$$\overline{\mathcal{M}}_{1,2} \times \overline{\mathcal{M}}_{0,3}$$



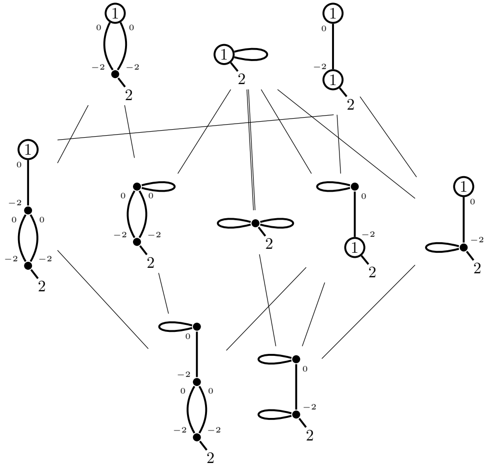
$$\overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{0,3} \times \overline{\mathcal{M}}_{0,3}$$

A **level graph** is a stable graph with a full order, equality permitted. Usually given by a *normalized level function*

$$\mathrm{lev}\colon \Gamma \to \{0, -1, \ldots, -N\}\,,$$

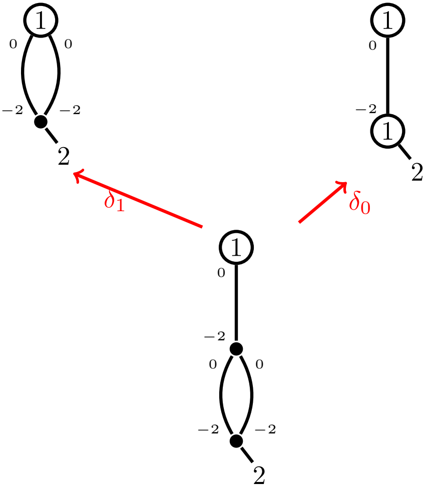
In pictures: Level zero = top level = top row of the stable graph.

The boundary of $\mathbb{P}\Xi\mathcal{M}_{2,1}(2)$



Adjacency of level graphs is by squishing a level passage.

This gives degeneration maps δ_I, indexed by the level passages that *remain*.


1
0
0
−2
−2
2
1
0
−2
1
2
δ1
δ0
1
0
−2
0
0
−2
−2
2

A **twisted differential** of type μ on a stable curve $(X, \mathbf{z})$ with dual graph Γ is a collection of meromorphic differentials $\eta_v \neq 0$ for $v \in V(\Gamma)$ such that the following conditions hold:

(0) **Vanishing as prescribed**

(1) **Matching orders at nodes**

$$\mathrm{ord}_{q_1} \eta_{v_1} + \mathrm{ord}_{q_2} \eta_{v_2} \;=\; -2\,.$$

(2) **Matching residues at simple poles**

A twisted differential is **compatible** with the level graph $(\Gamma, \succcurlyeq)$ if the following conditions hold:

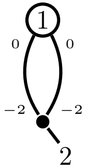
(3) **(Partial order)**

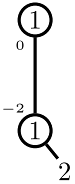
$$v_1 \succcurlyeq v_2 \quad \text{if and only if} \quad \operatorname{ord}_{q_1} \eta_{v_1} \geq -1$$

(4) **(Global residue condition)**

$\mathbb{P}\Xi\mathcal{M}_{1,2}(0,0)$

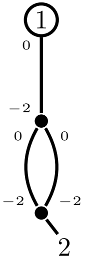
$\mathbb{P}\Xi\mathcal{M}_{0,3}(-2,-2,2)$





$\mathbb{P}\Xi\mathcal{M}_{1,1}(0)$

$\mathbb{P}\Xi\mathcal{M}_{1,2}(-2,2)$



$\mathbb{P}\Xi\mathcal{M}_{1,1}(0)$

$\mathbb{P}\Xi\mathcal{M}_{0,3}(-2,0,0)$

$\mathbb{P}\Xi\mathcal{M}_{0,3}(-2,-2,2)$

Define the incidence variety compactification $\mathrm{IVC}(\mu) \subset \overline{\mathcal{M}}_{g,n}$ to be the closure of the stratum $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ (with labeled zeros).

Theorem (BCGGM, 2016) Points in $\mathrm{IVC}(\mu)$ are characterized by the existence of a twisted differential compatible with some level graph structure on the dual graph of the underlying stable curve.

(A *twisted canonical divisor* in [Farkas-Pandharipande] does not include GRC. This chacterises a subspace of $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ with extra components with generic point in the boundary.)

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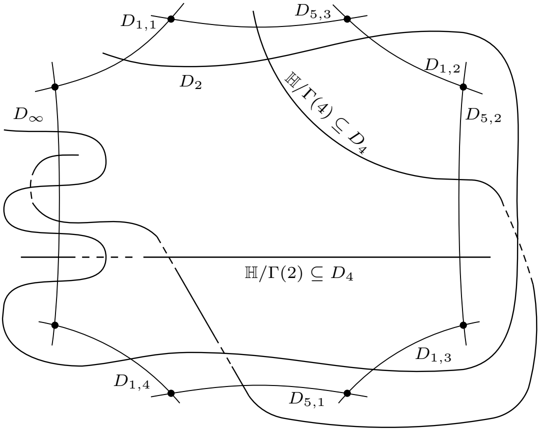
Problems of the notion twisted differential:

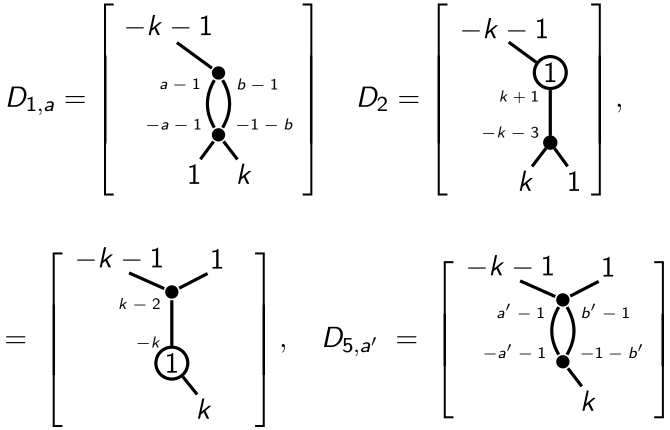
1) Need to separate branches:
   Enhanced level graphs and prong matchings
2) Parametrize differentials rescaled by level.
   Notation: $\eta_{(i)}$ the collection of all differentials on level i.

Solution: .... along with $\mathbb{P}\Xi\mathcal{M}_{1,3}(k, 1, -k-1)$



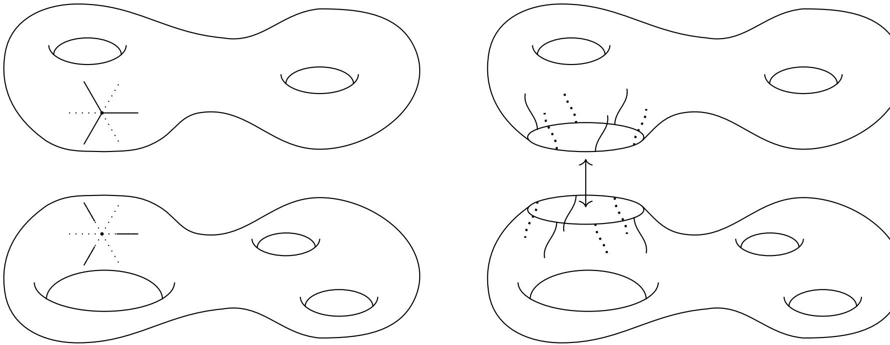
(Case $k = 5$)

Boundary divisors (besides D_h and some compact type D_3)



Schematic picture of a prong matching:

for a zero of order two with a pole of order four:



The numer of prongs is $\kappa = 3$.

Aim: compare the open boundary divisor $D^\circ_{1,a} = D_{1,a} \setminus \{\text{degenerations}\}$ to the product of the top level stratum $\mathbb{P}\Omega\mathcal{M}_{0,3}(a-1, b-1, -k-1)$ and the bottom level stratum $\mathbb{P}\Omega\mathcal{M}_{0,4}(k, 1, -a-1, -b-1)$:

- Fix a differential ω_0 in $\Omega\mathcal{M}_{0,3}(a-1, b-1, -k-1)$ on top level and take a quotient mod $\mathbb{C}^*$ for all level simultaneously in the end.
- Fix also a differential ω_{-1} in $\Omega\mathcal{M}_{0,4}(k, 1, -a-1, -b-1)$ on bottom level.
- For a node q a **prong matching** is an identification σ_q of the tangent spaces at the two ends $q^\pm$, mapping incoming horizontal prongs at q^+ to outgoing horizontal prongs at q^-.

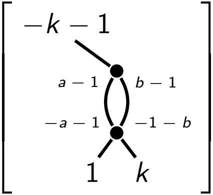
- The universal cover $\mathbb{C} \to \mathbb{C}^*$ of the rescaling torus acts simultaneously on $(\omega_{-1}, \sigma_{q_1}, \sigma_{q_2})$.
- The **prong rotation subgroup** $R_{D_{1,a}} \cong \mathbb{Z} \subset \mathbb{C}$ fixes ω_{-1}, thus acts on $(\sigma_{q_1}, \sigma_{q_2})$ only.
- The $R_{D_{1,a}}$-orbits are **prong matching equivalence classes**. There are $g_{1,a} = \gcd(\kappa_1, \kappa_2) = \gcd(a, b)$ equivalence classes.
- The subgroup $\mathrm{Tw}_{1,a} \subset R_{D_{1,a}}$ fixing all prongs matchings is called the **Twist group**. It acts trivially on $(\omega_{-1}, \sigma_{q_1}, \sigma_{q_2})$.
- The action of $\mathbb{C}$ on $(\omega_{-1}, \sigma_{q_1}, \sigma_{q_2})$ factors through the **level rotation torus** $T_{1,a} = \mathbb{C}/\mathrm{Tw}_{1,a}$.

First conclusion: The map

$$D^{\circ}_{1,a} \to \mathbb{P}\Omega\mathcal{M}_{0,3}(a-1, b-1, -k-1) \times \mathbb{P}\Omega\mathcal{M}_{0,4}(k, 1, -a-1, -b-1)$$

is a (usually connected) cover of degree $g_{1,a}$ (number of prong matching equivalence classes)

$D_{1,a} =$

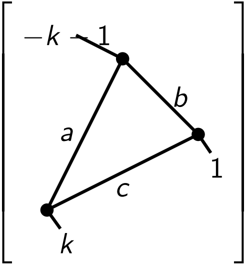


Construction:

- Complete the (level rotation) torus bundle $T_{1,a} \cong \mathbb{C}^*$-bundle over $D^\circ_{1,a}$ to a $\mathbb{C}$-bundle.
- The zero section is isomorphic to $D^\circ_{1,a}$.
- Over the complement of the zero section we construct a family of (ordinary) Abelian differentials in the stratum $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$.
- (This is the plumbing construction.)

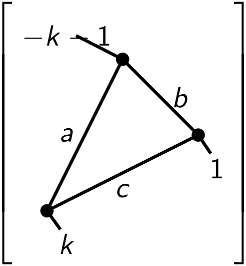
Degenerating further: Intersection of $D_{1,a}$ and $D_{5,a'}$

$D_\Delta =$



Here $c = b - 1$.
The universal covering $\mathbb{C}^2 \to (\mathbb{C}^*)^2$ acts by rescaling $(\omega_{-1}, \omega_{-2}, \sigma_1, \sigma_2, \sigma_3)$.

$D_\Delta =$

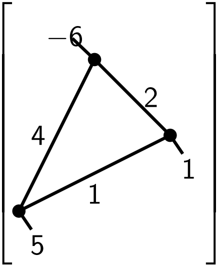


The prong rotation group $R_\Delta \cong \mathbb{Z}^2$ acts on $(\sigma_1, \sigma_2, \sigma_3)$.
The twist group Tw_Δ contains

$$\mathrm{Tw}^s_\Delta = \Big\langle (\mathrm{lcm}(a,b),0), (0, \mathrm{lcm}(a,c)) \Big\rangle$$

the **simple twist group**.

$D_\Delta =$



But if e.g. $k = 5$ and $b = 2$ so $a = 4$ and $c = 1$, then

$$[\mathrm{Tw}_\Delta : \mathrm{Tw}^s_\Delta] \;=\; 2$$

generated by $(2, -2)$.

This factor group causes a stack structure at the boundary not stemming from automorphisms of stable curves.

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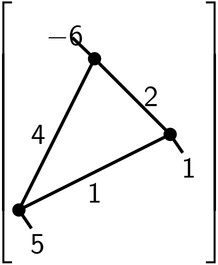
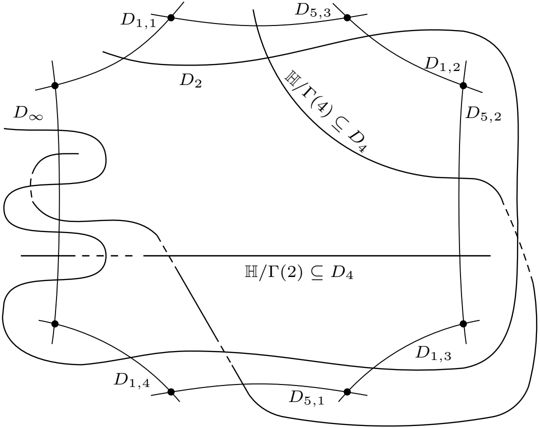


The action of $\mathbb{C}^2$ factors through the level rotation torus $T_\Delta = \mathbb{C}^2/\mathrm{Tw}_\Delta$.
It is covered by the **simple level rotation torus** $T_\Delta^s = \mathbb{C}^2/\mathrm{Tw}_\Delta^s$.

Construction II:

- Complete the (simpl level rotation) torus $T^s_{1,a} \cong (\mathbb{C}^*)^2$-bundle over D_Δ to a $\mathbb{C}^2$-bundle.
- The zero section is isomorphic to D_Δ.
- The locus where either coordinate is zero gives $D_{1,4}$ resp. $D_{5,1}$. (Normal crossing boundary divisor)
- Over the complement of the zero section we construct a family of (ordinary) Abelian differentials in the stratum $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$.
- (This is the plumbing construction.)
- In a neighborhood $\mathrm{Tw}_\Delta/\mathrm{Tw}^s_\Delta$ acts and the orbits are isomorphic multi-scale differentials.

Second conclusion: The intersection point of $D_{1,4}$ and $D_{5,1}$ has an order 2 quotient stack structure that does not come from automorphisms of stable curves.



Point-wise definition:

A **multi-scale differential of type** μ on a stable curve X is

(i) an enhanced level structure on the dual graph Γ of X,

(ii) a twisted differential of type μ compatible with the enhanced level structure,

(iii) and a prong-matching for each node of X joining components of non-equal level.

Two multi-scale differentials are equivalent if they differ by the action of the level rotation torus.

Missing steps/details in this story:

- Definition of the functor?
- Pullback of families via a morphism that factors through a boundary stratum?
- How to associate functorially a prong-matching equivalence class to a degenerating family?
- Recaling parameters, rescaling ensemble!

Tautological ring of $\overline{\mathcal{M}}_{g,n}$

The **tautological ring** is the smallest system of $\mathbb{Q}$-subalgebras $R^\bullet(\mathcal{M}_{g,n}) \subset \mathrm{CH}^\bullet(\mathcal{M}_{g,n})$ which

- contains the ψ-classes attached to the marked points,
- is closed under the pushfoward of the map forgetting a point, and
- is closed under the clutching homomorphisms induced by $\overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_1,n_1+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}$ and $\overline{\mathcal{M}}_{g,n+2} \to \overline{\mathcal{M}}_{g+1,n}$.

It contains the λ-classes and κ-classes.

Theorem (Graber-Pandharipande, 2001) The tautological ring of $\overline{\mathcal{M}}_{g,n}$ has a finite set of additive generators given by monomials in ψ-classes and κ-classes on every boundary stratum.

Tautological ring of $\mathbb{P}\Xi\mathcal{M}_{g,n}(\mu)$

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- contains the ψ-classes attached to the marked points,
- is closed under the pushfoward of the map forgetting a regular marked point (a zero of order zero), and
- is closed under the clutching homomorphisms $\zeta_{\Gamma,*}p^{[i],*}$ for all level graphs **without horizontal nodes**.

Theorem (CMZ, 2020) The tautological ring of $\overline{B} = \mathbb{P}\Xi\mathcal{M}_{g,n}(\mu)$ has a finite set of additive generators given by products of monomials of ψ-classes on each level of every boundary stratum without horizontal nodes.

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It contains

- the classes $\xi_\Gamma^{[i]}$ given by the first Chern class of the tautological line bundle at level i on the boundary stratum D_Γ, for any level graph Γ,
- and the κ-classes

We denote by $\mathrm{LG}_L(B)$ the set of level graphs without horizontal nodes and with $L + 1$ levels.

They define boundary strata of codimension L.

Ingredients:
Excess intersection formula ([CMZ20]), the relation (Sauvaget)

$$\xi \;=\; (m_i+1)\psi_{(i)} \;-\; \sum_{\Gamma\in\, _{(i)}\mathrm{LG}_1(\overline{B})} \ell_\Gamma [D_\Gamma]$$

Theorem (CMZ, 2020) Suppose that D_Γ is a divisor corresponding to a level graph $\Gamma \in \mathrm{LG}_1(\overline{B})$. Then

$$c_1(\mathcal{N}_\Gamma) = \frac{1}{\ell_\Gamma}\left(-\xi_\Gamma^\top - c_1(\mathcal{L}_\Gamma^\top) + \xi_\Gamma^\perp\right) \quad \text{in} \quad \mathrm{CH}^1(D_\Gamma)\,.$$

Here

$$\mathcal{L}^\top = \mathcal{O}\Big(\sum_{\Gamma\in\mathrm{LG}_1(B^\top)} \ell_\Gamma D_\Gamma\Big)\,.$$

Current knowledge about the topology of strata of meromorphic abelian differentials:

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- Homology groups: Nothing.
- Chern classes and Euler characteristics: **[CMZ20]**.

Some Euler characteristics

μ	$\chi(B)$
(0)	$-\frac{1}{12}$
(2)	$-\frac{1}{40}$
$(1,1)$	$\frac{1}{30}$
(4)	$-\frac{55}{504}$
$(3,1)$	$\frac{16}{33}$
$(2,2)$	$\frac{15}{56}$
$(2,1,1)$	$-\frac{6}{7}$
$(1,1,1,1)$	$\frac{11}{3}$

μ	$\chi(B)$
(6)	$-\frac{1169}{720}$
$(5,1)$	$\frac{27}{5}$
$(4,2)$	$\frac{76}{15}$
$(3,3)$	$\frac{188}{45}$
$(4,1,1)$	$-\frac{200}{9}$
$(3,2,1)$	$-\frac{96}{5}$
$(2,2,2)$	$-\frac{187}{10}$
$(2,2,1,1)$	$\frac{504}{5}$

Strategy for $\mathcal{M}_{g,n}$.

Find a complex (**the arc complex**) on which the fundamental group acts and count orbits and stabilizer groups:

Theorem (Harer-Zagier, 1986) The Euler characteristic of the moduli space of curves is given by

$$\chi(\mathcal{M}_{g,1}) = -B_{2g}/2g \quad \text{and} \quad \chi(\mathcal{M}_{g,n+1}) = (2-2g-n)\chi(\mathcal{M}_{g,n})$$

Strategy for $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$.

Show that the **Chern classes of the logarithmic tangent bundle of $\overline{B}$ are tautological** and compute them in terms of additive generators.

Main tool: An **Euler sequence** for compactified strata.

Recall that if X is a compact complex manifold of dimension n, then

$$\chi(X) = \sum_{i \geq 0} (-1)^i \dim H^i(X, \mathbb{Q}).$$

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By Hirzebruch-Riemann-Roch we can compute it via the Gauß-Bonnet formula

$$\chi(X) = \int_X \mathrm{c}_n(X)$$

where $\mathrm{c}_n(X) = \mathrm{c}_n(\mathcal{T}_X)$ is the top Chern class of the tangent bundle of X.

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1. If X is not compact, but there is a compact manifold $\overline{X}$ such that $X \subset \overline{X}$ is an open submanifold and $D = \overline{X} \setminus X$ is a smooth normal crossing divisor, then

$$\chi(X) = \int_{\overline{X}} c_n(\mathcal{T}_{\overline{X}}(-\log(D)))$$

where $\mathcal{T}_{\overline{X}}(-\log(D))$ is the dual of the logarithmic cotangent bundle.

We can generalize the Gauß-Bonnet formula in two ways:

1. If X is not compact, but there is a compact manifold $\overline{X}$ such that $X \subset \overline{X}$ is an open submanifold and $D = \overline{X} \setminus X$ is a smooth normal crossing divisor, then

$$\chi(X) = \int_{\overline{X}} c_n(\mathcal{T}_{\overline{X}}(-\log(D)))$$

where $\mathcal{T}_{\overline{X}}(-\log(D))$ is the dual of the logarithmic cotangent bundle.

2. If X is an orbifold/analytic Deligne-Mumford stack, then

$$\chi^{\mathrm{orb}}(X) = \int_{\overline{X}} c_n(\mathcal{T}_{\overline{X}}(-\log(D)))$$

still holds by replacing with orbifold Euler characteristic.

Logarithmic (co)-tangent bundle

If $\overline{X}$ is a compact manifold with a normal crossing divisor D. Locally, in coordinates $x_1, \ldots, x_{\dim(\overline{X})}$, if $D = \left\{ \prod_{i=1}^{s} x_i = 0 \right\}$, then

$$\Omega^1_{\overline{X}}(\log D) \;=\; \left\langle \frac{dx_1}{x_1}, \ldots, \frac{dx_s}{x_s}, dx_{s+1}, \ldots, dx_{\dim(\overline{X})} \right\rangle .$$

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Orbifold Euler characteristic

If $X = Y/\Gamma$ is a global quotient of a complex space Y and Δ is a finite-index torsion-free subgroup, then

$$\chi^{orb}(X) := \frac{1}{[\Gamma : \Delta]} \chi(Y/\Delta)$$

If X is not a global quotient, apply this mechanism locally.

Example: $\mathcal{M}_{1,1} = \mathbb{P}\Omega\mathcal{M}_{1,1}(0)$

Since $B = \mathcal{M}_{1,1} = \mathbb{H}/\operatorname{SL}_2(\mathbb{Z})$, take a complement of $\pm I_2$ in $\Gamma(2)$, i.e. such that $\Gamma(2) = \Gamma(2)^+ \times \{\pm I_2\}$. Then

$$[\operatorname{SL}_2(\mathbb{Z}) : \Gamma(2)^+] = 12 \quad \text{and} \quad \mathbb{H}/\Gamma(2)^+ \cong \mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$$

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On the other hand

$$\int_{\overline{\mathcal{M}}_{1,1}} c_1(\Omega^1(\log(\infty))) = \frac{1}{2\pi}\int_{\overline{\mathcal{M}}_{1,1}} d\,\mathrm{vol}_{hyp} = \frac{1}{12}$$

Theorem (CMZ, 2020) The Chern character of the logarithmic cotangent bundle is tautological and given by

$$\mathrm{ch}(\Omega^1_{\overline{B}}(\log D)) = e^{\xi} \cdot \sum_{L=0}^{N-1} \sum_{\Gamma \in \mathrm{LG}_L(B)} \ell_\Gamma \left(N - N^T_{\delta_L(\Gamma)} \right) \mathfrak{i}_{\Gamma *} \Big(\prod_{i=1}^{L} \mathrm{td} \left(\mathcal{N}^{\otimes -\ell_{\Gamma,i}}_{\Gamma/\delta_i^{\complement}(\Gamma)} \right)^{-1} \Big) ,$$

where $\mathcal{N}$ denotes the normal bundle, δ_L and $\delta_i^{\complement}$ are the undegeneration maps of the boundary strata, and $N = \dim(\overline{B}) + 1$ and $N^T_{\delta_L(\Gamma)} = \dim(B^T_{\delta_L(\Gamma)}) + 1$ are the unprojectivised dimensions.

Corollary (CMZ, 2020) The canonical class of the moduli space $\overline{B} = \mathbb{P}\Xi\mathcal{M}_{g,n}(\mu)$ of multi-scale differentials is

$$c_1(\Omega^1_{\overline{B}}(\log D)) = N \cdot \xi + \sum_{\Gamma \in \mathrm{LG}_1(\mathrm{B})} (N - N_\Gamma^\top)\ell_\Gamma D_\Gamma \qquad \in \mathrm{CH}^1(\overline{B})\,.$$

Compare:

Theorem (Harris-Mumford, 1983) The canonical class of the moduli stack of curves $\overline{\mathcal{M}}_g$ is

$$c_1(\Omega^1_{\overline{\mathcal{M}}_g}) = 13\lambda - 2\sum_{i=0}^{\lfloor g/2 \rfloor} \delta_i \,.$$

Theorem (CMZ, 2020) The Euler characteristic of the moduli space $B = \mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ is given by

$$\chi(B) = (-1)^d \sum_{L=0}^{d} \sum_{\Gamma \in \mathrm{LG}_L(B)} \ell_\Gamma N_\Gamma^\top \int_{\overline{B}} \prod_{i=0}^{L-1} (\xi_\Gamma^{[i]})^{d_\Gamma^{[i]}}$$

where $d_\Gamma^{[i]}$ is the dimension of the projectivized moduli space at level i and $N_\Gamma^\top = d_\Gamma^{[0]} + 1$.

How to evaluate this:

- **Construct all possible non-horizontal enhanced level graphs**: construct all 2-level graphs and recursively glue the 2-level graphs of each level stratum.

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How to evaluate this:

- **Construct all possible non-horizontal enhanced level graphs**: construct all 2-level graphs and recursively glue the 2-level graphs of each level stratum.
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  For all other strata, the evaluation of tautological classes is performed using the formula for fundamental classes of strata conjectured in [FP18] and [Sch18] and proven recently in [HS19] and [BHPSS20].

The evaluation of these formulas is performed by a `sage` package `diffstrata` that builds on the package `admcycles` for computation in the moduli space of curves ([DSZ20]).

μ	$(0^2,-2)$	$(2,-2)$	$(1^2,-2)$	$(4,-2)$	$(3,1,-2)$	$(2,1,-3)$	$(5,-3)$
$\int_{\overline{B}} \xi^{\dim(B)}$	1	$-\frac{1}{8}$	0	$-\frac{-23}{1152},$	0	$\frac{5}{8}$	$-\frac{21}{20}$

μ	$\chi(B)$
(0)	$-\frac{1}{12}$
(2)	$-\frac{1}{40}$
(1^2)	$\frac{1}{30}$
(4)	$-\frac{55}{504}$
$(3,1)$	$\frac{16}{33}$
(2^2)	$\frac{15}{56}$
$(2,1^2)$	$-\frac{6}{7}$
(1^4)	$\frac{11}{3}$

μ	$\chi(B)$
(6)	$-\frac{1169}{720}$
$(5,1)$	$\frac{27}{5}$
$(4,2)$	$\frac{76}{15}$
(3^2)	$\frac{188}{45}$
$(4,1^2)$	$-\frac{200}{9}$
$(3,2,1)$	$-\frac{96}{5}$
(2^3)	$-\frac{187}{10}$
$(2^2,1^2)$	$\frac{504}{5}$

μ	$\chi(B)$
$(4,-2)$	$-\frac{19}{24}$
$(4,-1^2)$	$\frac{8}{5}$
$(3,1,-2)$	$-\frac{28}{15}$
$(3,1,-1^2)$	-4
$(2^2,-2)$	$\frac{17}{10}$
$(2^2,-1^2)$	-4
$(2,1^2,-2)$	-6
$(2,1^2,-1^2)$	14

Idea of proof of the formulas: use the same strategy as for projective space!

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Suppose $B = \mathbb{P}(V)$ is the projective space of a vector space V of dimension N. Then there is an exact sequence, the **Euler sequence**

$$0 \longrightarrow \Omega^1_{\mathbb{P}(V)} \longrightarrow V \otimes \mathcal{O}_{\mathbb{P}(V)}(-1) \xrightarrow{\text{ev}} \mathcal{O}_{\mathbb{P}(V)} \longrightarrow 0\,.$$

From this one gets

$$\mathrm{ch}(\Omega^1_{\mathbb{P}(V)}) = N(1 + \xi_{\mathbb{P}(V)})$$

$$c_k(\mathbb{P}(V)) = \binom{N}{k} \xi^k_{\mathbb{P}(V)}$$

Open stratum:

Projective strata have a projective structure induced by period coordinates. Glue together all local Euler sequences!

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Projective strata have a projective structure induced by period coordinates. Glue together all local Euler sequences!

Suppose $B = \mathbb{P}\Omega\mathcal{M}_{g,n}(\mu)$ is an (open) stratum. Then there is an exact sequence

$$0 \longrightarrow \Omega^1_B \longrightarrow (\mathcal{H}^1_{\mathrm{rel}})^\vee \otimes \mathcal{O}_B(-1) \xrightarrow{\mathrm{ev}} \mathcal{O}_B \longrightarrow 0$$

where $\mathcal{H}^1_{\mathrm{rel}}$ is the local system with fiber $V = \mathrm{H}^1(X \setminus P, Z; \mathbb{C})$ and where

$$\mathrm{ev}\colon (\mathcal{H}^1_{\mathrm{rel}})^\vee \otimes \mathcal{O}_B(-1) \to \mathcal{O}_B, \quad \gamma \otimes \omega \mapsto \int_\gamma \omega$$

Over the compactification: extend and correct an error term.
There is an exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow (\overline{\mathcal{H}}^1_{\mathrm{rel}})^\vee \otimes \mathcal{O}_{\overline{B}}(-1) \xrightarrow{\mathrm{ev}} \mathcal{O}_{\overline{B}} \longrightarrow 0\,,$$

where $\mathcal{H}^1_{\mathrm{rel}}$ is the Deligne extension of the local system and where

$$0 \longrightarrow \Omega^1_{\overline{B}}(\log D) \otimes \mathcal{L}^{-1} \to \mathcal{K} \to \mathcal{C} \longrightarrow 0$$

with $\mathcal{C}$ and $\mathcal{L}$ supported on the non-horizontal boundary divisor D.

A local calculation near the boundary implies that $\mathcal{C} = \oplus_{\Gamma \in \mathrm{LG}_1(\mathrm{B})} \mathcal{C}_\Gamma$ with

$$\mathrm{ch}(\mathcal{C}_\Gamma) \; = \; \mathrm{ch}\Big((\mathrm{i}_\Gamma)_* \big(\bigoplus_{j=0}^{\ell_\Gamma - 1} \mathcal{N}_{D_\Gamma}^{\otimes -j} \otimes \Omega^1_{B_\Gamma^\top}(\log(D_\Gamma^\top)) \otimes \mathcal{L}_{B_\Gamma^\top}^{-1} \big) \Big)$$

Via Grothendieck-Riemann-Roch, we get a recursive expression for $\mathrm{ch}(\Omega^1_B(\log(D))$ that we can eventually expand.

Thank you!