Turán numbers for a 4-uniform hypergraph

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Based on joint work with
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Turán numbers

For n, r and an r-uniform hypergraph $\mathcal{H}$, the *Turán number* is

$$\mathrm{ex}(n, \mathcal{H}) = \text{max. number of hyperedges in } r\text{-unif.,} \\ \mathcal{H}\text{-free hypergraph on } n \text{ vertices.}$$

Turán density for $\mathcal{H}$:

$$\pi(\mathcal{H}) = \lim_{n \to \infty} \frac{\mathrm{ex}(n, \mathcal{H})}{\binom{n}{r}}$$

- (Case $r = 2$) Erdős-Stone-Simonovits: $\pi(H) = 1 - \frac{1}{\chi(H)-1}$
- Exact Turán numbers known for many classes of graphs
- Much less known for hypergraphs

Turán density for K_4^-

K_4^- = 3-uniform hypergraph on 4 vertices with 3 edges

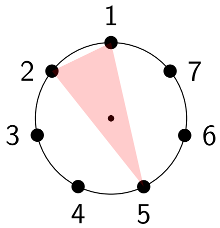
$$0.2857 \approx \frac{2}{7} \leq \pi(K_4^-) \leq 0.2871$$

- Upper bound: Baber and Talbot (2011) using flag algebra techniques
- Lower bound: Frankl and Füredi (1984) construction based on a fixed hypergraph with 6 vertices and 10 hyperedges

4-sets have 0 or 2 hyperedges

Frankl and Füredi (1984):
Characterized all 3-uniform hypergraphs with the ppty that any 4 vertices contains either 0 or 2 hyperedges as one of two classes of hypergraphs.

Example 1: Place n vertices around a circle, no two on a line through the center. Hyperedges = sets of three vertices whose convex hull contains the center; $\leq \frac{1}{4}\binom{n+1}{3}$ edges.



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135	246	357
136	247	
145	256	367
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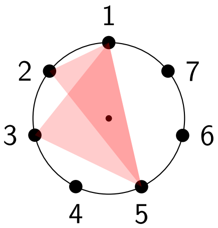

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Example 2: Blow-up of the following 3-hypergraph with 6 vertices and 10 edges gives $\sim 10\left(\frac{n}{6}\right)^3 \approx \frac{5}{18}\binom{n}{3}$ edges with n vertices.

Vertices $= \{0, 1, 2, 3, 4, 5\}$
Edges:

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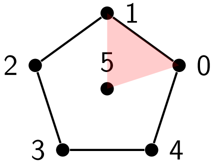
One construction:
Hyperedge = 3 vertices that contain one edge.



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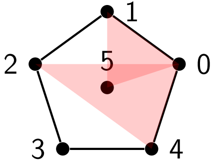

Two-graphs

Example 2 and the graph construction are an example of a *two-graph* (introduced by Higman):

Def: A *two-graph* is a 3-uniform hypergraph with the property that any 4-set contains an **even** number of edges.

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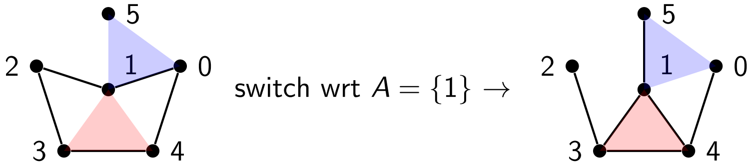
Def: A *two-graph* is a 3-uniform hypergraph with the property that any 4-set contains an **even** number of edges.

Construction: For any graph, define a hypergraph whose edges are the 3-sets that contain an odd number of edges. Result is always a two-graph.

Every two-graph arises in this way: Pick any vertex x and build a graph with all pairs $\{y, z\}$ contained in a hyperedge with x.

Graph switching

Given a graph $G = (V, E)$ and $A \subseteq V$, *switching w.r.t* A: interchange edges and non-edges between A and A^c.



Graphs G, H give the same two-graph iff one can be obtained from the other via switching.

Other sizes of hyperedges?

Question (Frankl, Füredi):
What is max. value of $e(\mathcal{H})$ among all r-uniform hypergraphs on n vertices with the ppty that every $r+1$ vertices span either 0 or 2 edges?

Construction:

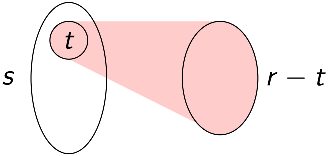
- Vertices $=$ n points on the surface of a sphere in $\mathbb{R}^{r-1}$.
- Edges $=$ r-sets whose convex hull contains the centre of the sphere.

Choosing position of vertices at random gives $\frac{(1+o(1))}{2^{r-1}}\binom{n}{r}$ edges.

Related problems

Bollobás, Leader, Malvenuto (2011): r-uniform (s, t)-*Daisy*

$\mathcal{D}_r(s, t)$:



A hypergraph is $\mathcal{D}_r(3, 2)$-free iff every $(r + 1)$-set has at most 2 edges.

Reiher, Rödl, Schacht (2017): Determine Turán density with additional ‘uniform density’ condition for r-uniform hypergraph on $r + 1$ vertices with 3 edges.

Case for 4-uniform hypergraphs

Question (Frankl, Füredi $r = 4$):
What is max. value of $e(\mathcal{H})$ among all 4-uniform hypergraphs on n vertices with the ppty that every 5 vertices span either 0 or 2 edges?

Sphere construction gives such a hypergraph with

$$\frac{(1 + o(1))}{8} \binom{n}{4}$$

edges.

Infinitely many answers for $r = 4$

Theorem (G., Semeraro)

For each prime power $q \equiv 3 \pmod 4$, there exists a 4-uniform hypergraph, $\mathcal{H}_q$, on $q+1$ vertices with the following properties:

- *any 5-set spans 0 or 2 edges,*
- $e(\mathcal{H}_q) = \frac{q+1}{16}\binom{q+1}{3}$, *and*
- *every 3-set is contained in exactly $\frac{q+1}{4}$ edges.*

Theorem (G., Semeraro)

For any prime power $q \equiv 3 \pmod 4$,

$$\mathrm{ex}(q+1, \{1234, 1235, 1245\}) = \frac{(q+1)}{16}\binom{q+1}{3}.$$

Upper bound

Lemma

If $\mathcal{H}$ is an r-uniform hypergraph on n vertices with the ppty that every $(r+1)$-set contains at most 2 hyperedges, then

$$e(\mathcal{H}) = |E(\mathcal{H})| \leq \frac{n}{r^2}\binom{n}{r-1}.$$

Proof sketch.
(Based on de Caen's 1983 bound for hypergraph Turán numbers.)
Double-count pairs (A, B) with

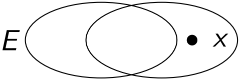
- $|A| = |B| = r$,
- $|A \cap B| = r - 1$, and
- $A \in \mathcal{H}$, $B \notin \mathcal{H}$.

Pf continued

$$* = |\{(A,B): |A| = |B| = r,\ |A \cap B| = r-1,\ A \in \mathcal{H} \text{ and } B \notin \mathcal{H}\}|$$

Lower bound:

For every edge E and $x \notin E$, $E \cup \{x\}$ contains ≤ 1 other edge.



There are at least $r - 1$ choices for $B \notin \mathcal{H}$ with $|B \cap E| = r - 1$.

$$* \geq e(\mathcal{H})(n-r)(r-1).$$

Upper bound:

For every $(r-1)$-set C, let $a_C = |\{A \in \mathcal{H} : \ C \subseteq A\}|$.
Since every edge has r different $(r-1)$-subsets:

$$\sum a_C = re(\mathcal{H}).$$

Num. of pairs (A, B) with $A \in \mathcal{H}$, $B \notin \mathcal{H}$ and $A \cap B = C$ is $a_C(n-r+1-a_C)$.

$$* = \sum a_C(n-r+1-a_c) = (n-r+1)re(\mathcal{H}) - \sum a_C^2$$
$$\leq (n-r+1)re(\mathcal{H}) - \frac{r^2 e(\mathcal{H})^2}{\binom{n}{r-1}}.$$

Thus,

$$e(\mathcal{H})(n-r)(r-1) \geq (n-r+1)re(\mathcal{H}) - \frac{r^2 e(\mathcal{H})^2}{\binom{n}{r-1}}.$$

Construction idea for $\mathcal{H}_q$

Start with

- Vertices $= \mathbb{F}_q$
- Edges: $\{a, b, c, d\}$ s.t. for every perm. $\sigma \in S_4$,

$$(\sigma(a) - \sigma(b))(\sigma(b) - \sigma(c))(\sigma(c) - \sigma(d))(\sigma(d) - \sigma(a))$$

  is a quadratic non-residue in $\mathbb{F}_q$.

Every 5-set spans 0 or 2 edges.

To count edges, let χ be the square character on $\mathbb{F}_q$ and use identities:

$$\sum_{x \in \mathbb{F}_q} \chi(x) = 0.$$

For every $y \neq 0$,

$$\sum_{x \in \mathbb{F}_q} \chi(x)\chi(x+y) = -1.$$

Total number of edges =

$$\frac{q-3}{16}\binom{q+1}{3} < \frac{q}{16}\binom{q}{3}.$$

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Extend to $\mathbb{P}^1\mathbb{F}_q$ with $\frac{1}{4}\binom{q+1}{3}$ edges containing the point at infinity to get the full hypergraph.

To be precise, replace $\chi(a-b)$ with $\chi\left(\begin{vmatrix} a & b \\ 1 & 1 \end{vmatrix}\right)$ and extend same construction with homogeneous coordinates.

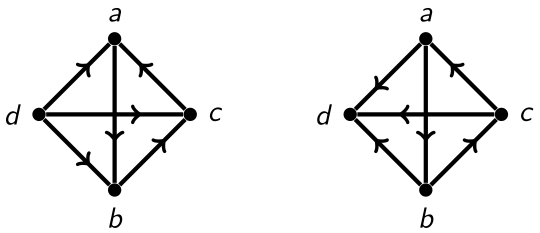
Turán density

Theorem (Baber)

$\pi(\{1234, 1235, 1245\}) = \frac{1}{4}$.

Proof.

For any tournament on n vertices, choose 4-sets that induce tournaments isomorphic to either of the following:



Choosing the tournament at random gives at least $\frac{1}{4}\binom{n}{4}$ hyperedges.

Connection to $\mathcal{H}_q$

Let $T(q)$ be the Paley tournament on $\mathbb{F}_q$:

- Vertices $= \mathbb{F}_q$
- Arcs $= a \to b$ iff $b - a$ is a quadratic residue.

Extend to $T^*(q)$: add a vertex 'at infinity' with all edges directed towards it.

Applying Baber's construction to $T^*(q)$ gives the hypergraph $\mathcal{H}_q$!

When $n \not\equiv 0 \pmod 4$

Belkouche, Boussaïri, Lakhlifi, Zaidi (2020):
If n is **odd** and $\mathcal{H}$ is a 4-uniform hypergraph on n vertices with the ppty that any 5 vertices contains either 0 or 2 hyperedges, then

$$e(\mathcal{H}) \leq \frac{(n+1)(n-3)}{16(n-2)}\binom{n}{3}.$$

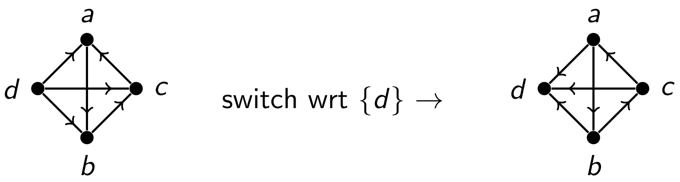
Using Baber's construction from tournaments, they show that the existence of certain skew-conference matrices would imply that the upper bounds are achieved.

Tournament switching

Definition

Tournaments T_1 and T_2 on the same vertex set V are *switching equivalent* iff $\exists\ A \subseteq V$ so that T_2 can be obtained from T_1 by reversing the orientation of all edges between A and $V \setminus A$.

The two tournaments:



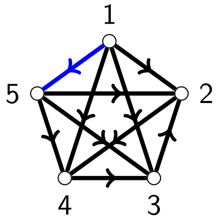
are a switching equivalence class.

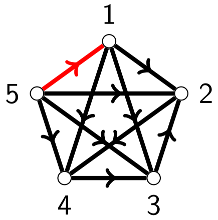
For any tournament T, let $\mathcal{H}_T$ be the associated hypergraph given by Baber's construction.

Lemma

If tournaments T and T' are switching equivalent, then $\mathcal{H}_T = \mathcal{H}_{T'}$.

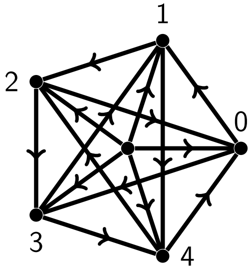
The converse fails: The following two tournaments are **not** switching equivalent, but both yield the same hypergraph $\{1234, 2345\}$.





6-uniform hypergraphs

The switching equivalence classes of the following tournament:



and constructions using $T^*(q)$ show that

$$\frac{9}{64} \leq \pi(\{123456, 123457, 123467\}) \leq \frac{1}{6}$$

and

$$\text{ex}(12, \{123456, 123457, 123467\}) = 264.$$

Not all extremal examples from tournaments

There are extremal examples that do not arise from tournaments. Hughes (1965) gave a $3-(12,4,3)$ design, $\mathcal{M}$, associated with the Mathieu group M_{11}.

Theorem (G, Semeraro)

The hypergraph $\mathcal{M}$ as the properties

- *any 5-set spans 0 or 2 edges,*
- $e(\mathcal{M}) = 165 = \frac{12}{16}\binom{12}{3}$, *and*
- *there is no tournament T with $\mathcal{H}_T \cong \mathcal{M}$.*

Question: Classification of 4-hypergraphs with ppty that every 5 vertices span 0 or 2 edges?

Thank you!