Sampling symmetric Gibbs distributions on sparse random graphs with contiguity

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Workshop on Inference Problems
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Gibbs distribution

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- configuration σ is assigned probability measure

$$\mu(\sigma) \propto \texttt{weight}(\sigma)$$

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- for $q = 2$ we have the Ising model
- for $\beta = -\infty$ we have the Colouring model

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- generate efficiently $\boldsymbol{\sigma}$ which is distributed "close" to μ
- focus on the range of parameters of μ in which we can get "good" approximate samples

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Sampling Problem on $G(n, m)$

- focus on approximate sampling
- use concepts from physics and related problems on random graphs to get better sampling algorithms

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Our approach has nothing to do with all the above ...

Example from the past

Example with the Colouring Model

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- from (Efthymiou 2012).

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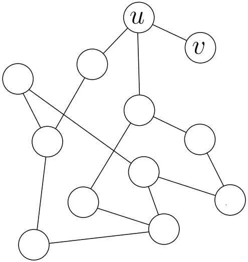

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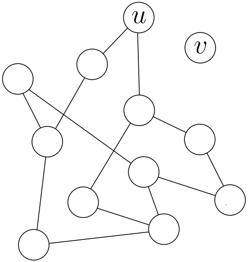

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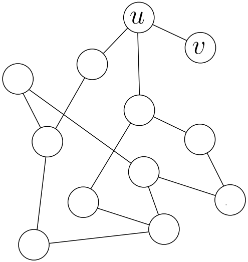

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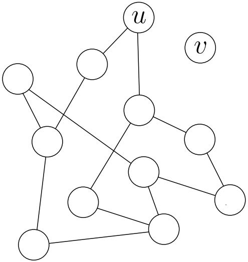

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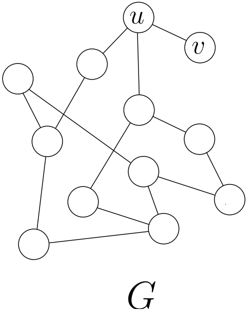

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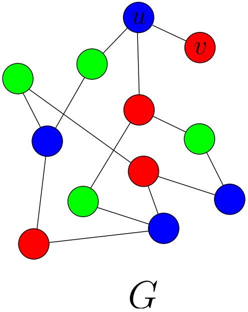

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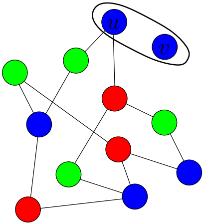

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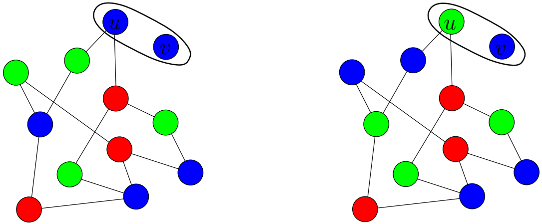

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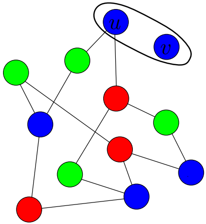

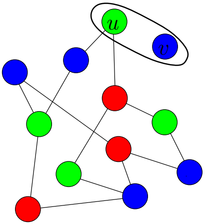

Be careful...

We can not change the colours of the vertices **arbitrarily**.

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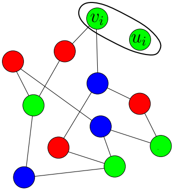

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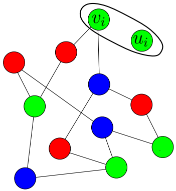

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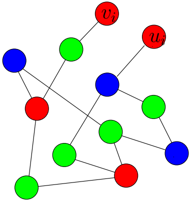

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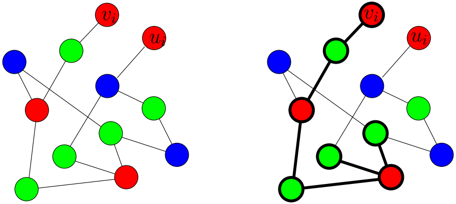

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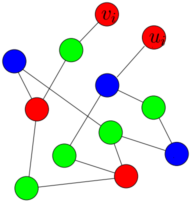

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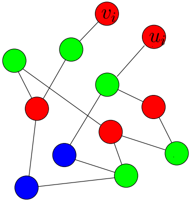

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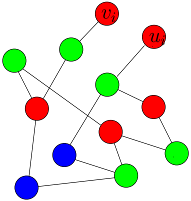

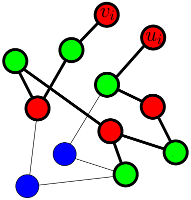

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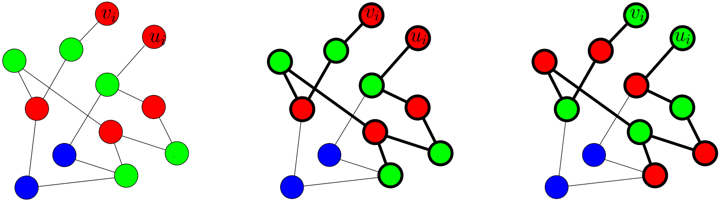

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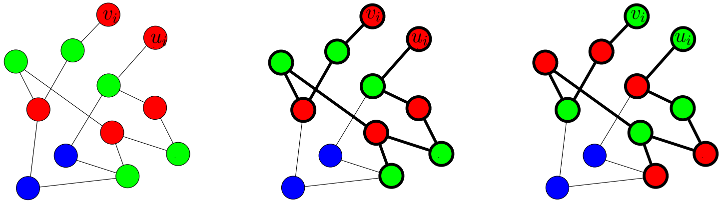

Failure

When both v_i and u_i change colour `Update` fails

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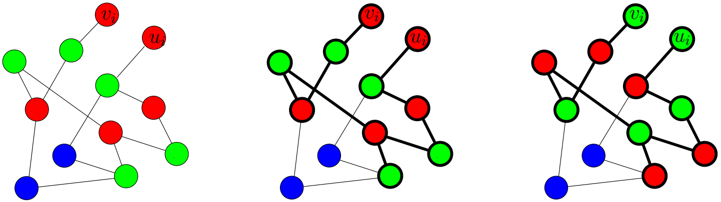

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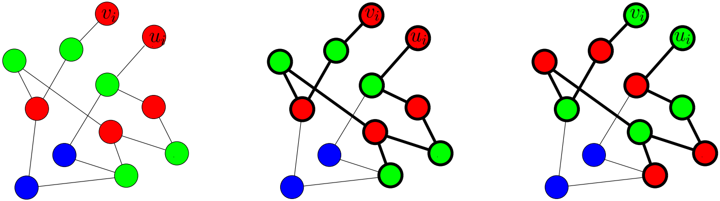

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- the output is approximately Gibbs distributed

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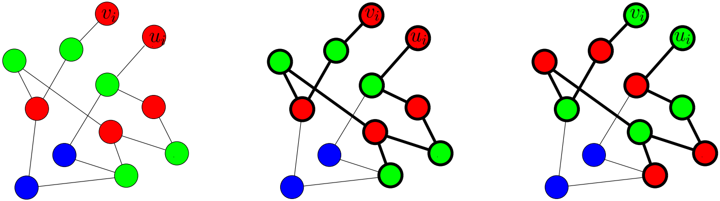

Error for `Update`

≈ the probability of failure

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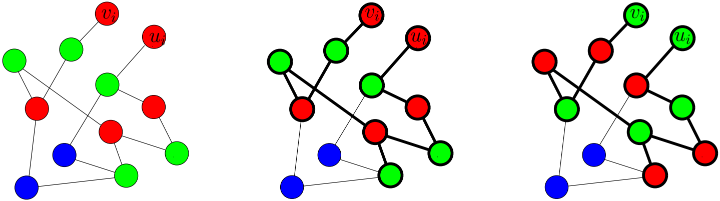

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The sampling algorithm that uses `Update` is approximation too

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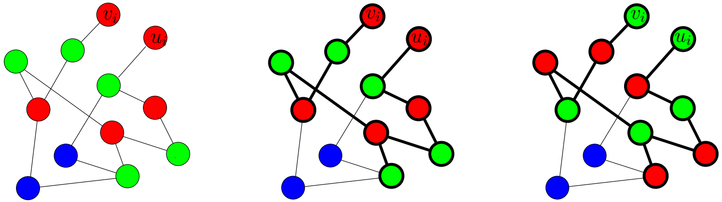

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$$\text{output error} \approx \text{there is a failure is some iteration}$$

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  - failure implies that we have an *extensive* chain
- care should be taken for v_i, u_i are at short distance
  - the update for such pairs is different (didn't show that)

Performance of Sampler

Theorem (Efthymiou 2016)

For $\epsilon > 0$, for large $d > 0$ and $q \geq (1+\epsilon)d$ we have the following: With probability $1 - o(1)$ over the instances of $\boldsymbol{G}(n, m)$ the algorithm generates a q-colouring of $\boldsymbol{G}$ which is distributed within total variation distance $n^{-\Omega(1)}$ from the q-colouring model.

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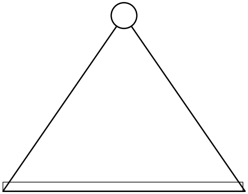

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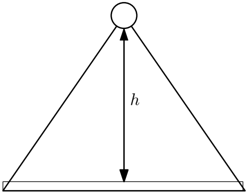

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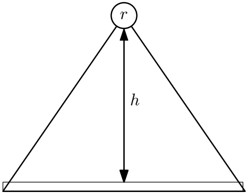

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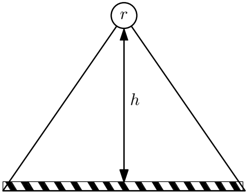

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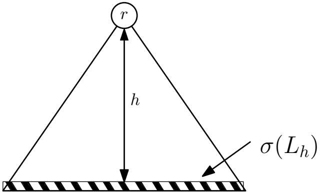

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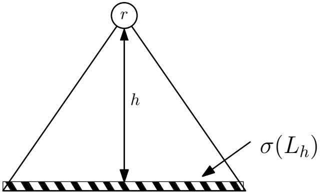

$$||\mu(\cdot) - \mu(\cdot \mid \sigma(L_h))||_{\{r\}}$$

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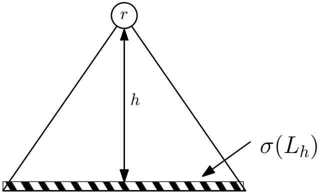

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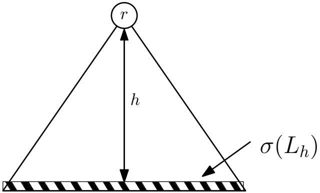

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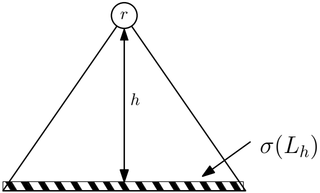

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Remark

The above are for both graphs and hypergraphs

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Objective

Generate efficiently $\boldsymbol{\tau}$ distributed close to μ'

Ideal Solution

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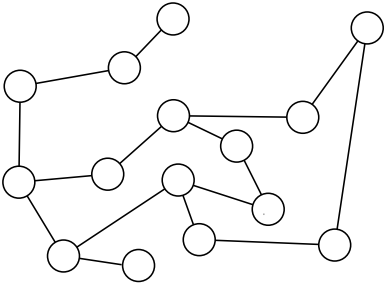

G

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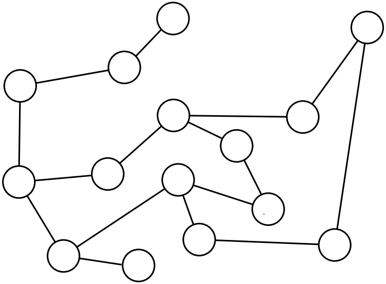
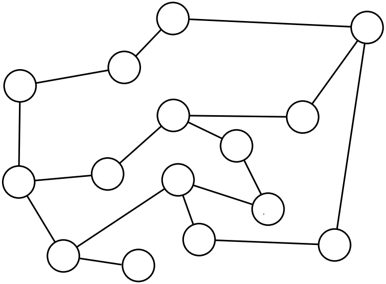


G



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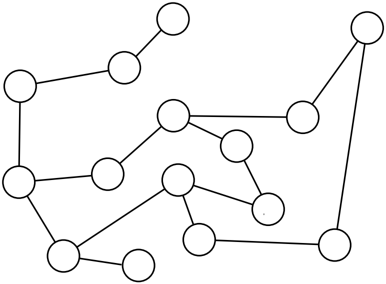

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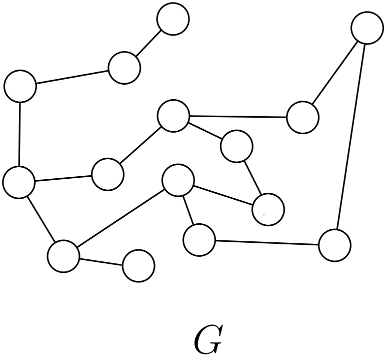

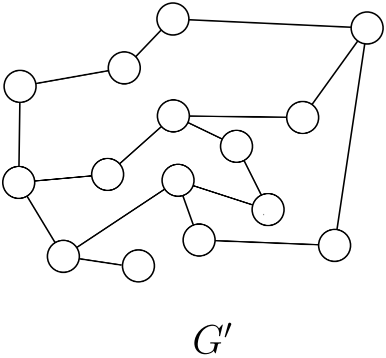

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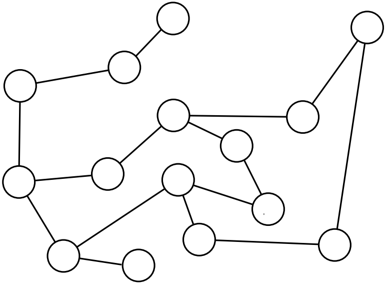

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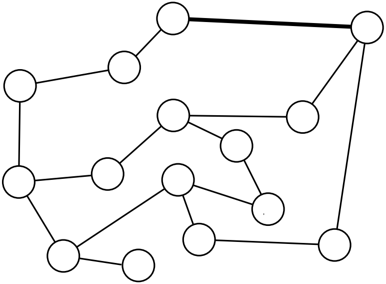

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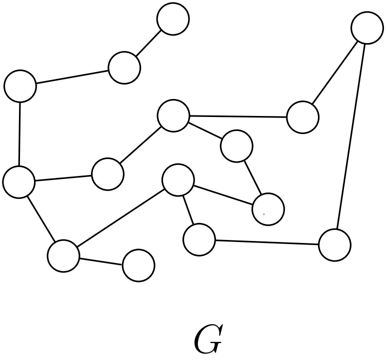

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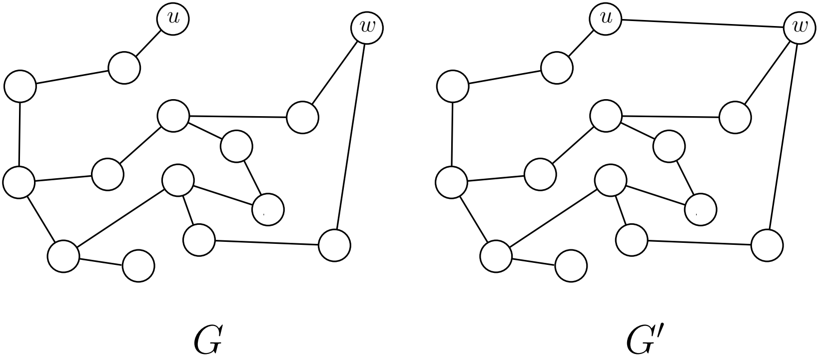

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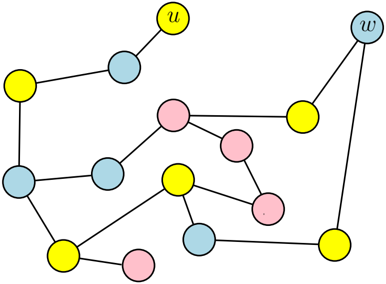

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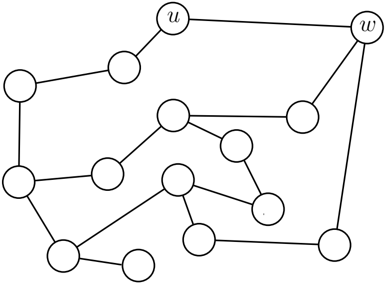

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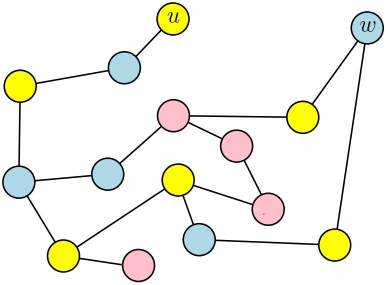

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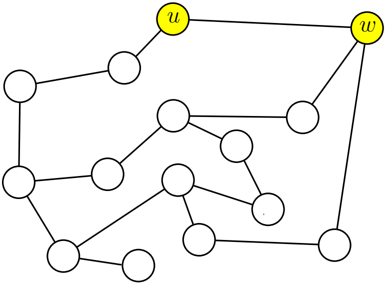

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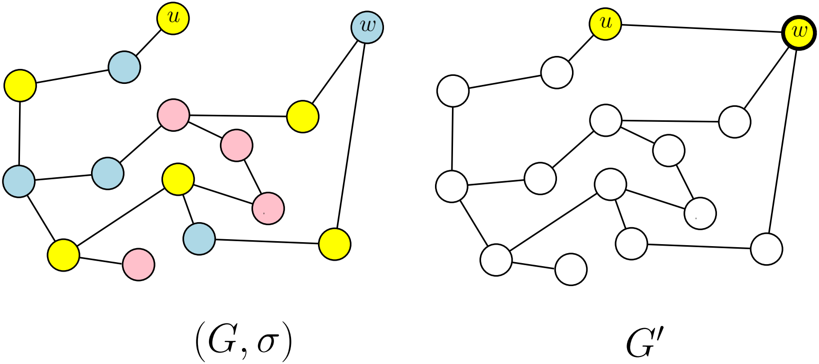

vertex w is a **disagreement** with spins $\{\texttt{blue}, \texttt{yellow}\}$

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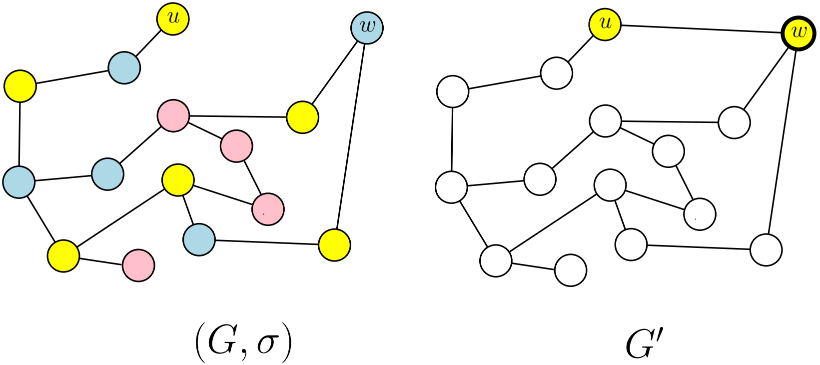

iteratively visit the vertices of G' one by one and decide their configuration at τ

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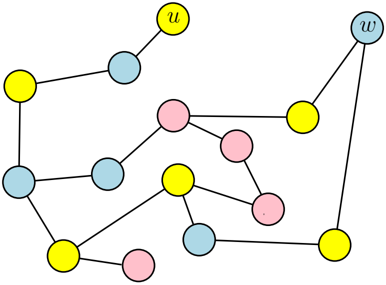

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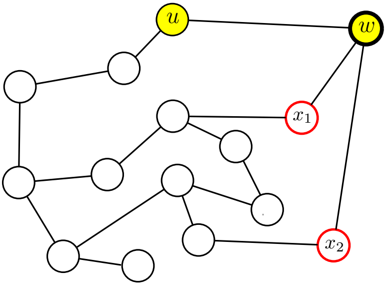

G'

look for z's, neighbours of w with $\sigma(z) \in \{\texttt{blue}, \texttt{yellow}\}$

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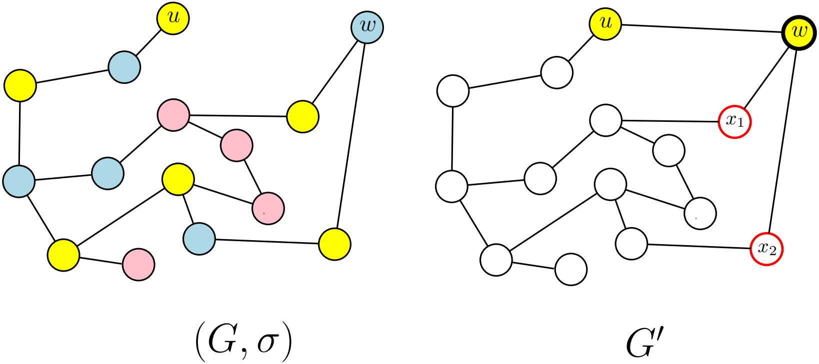

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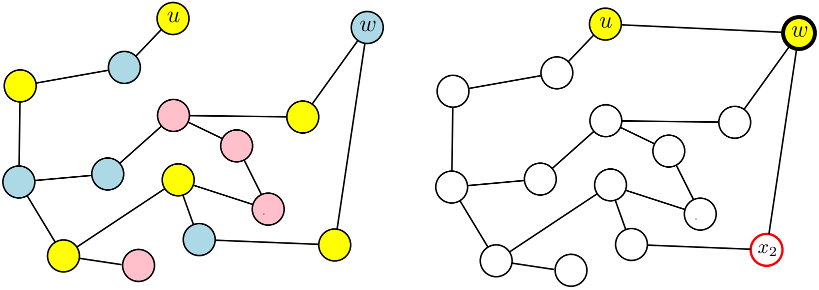

(G, σ) G'

pick x_2 and decide $\tau(x_2)$ such that $\tau(x_2) \in \{\texttt{blue}, \texttt{yellow}\}$

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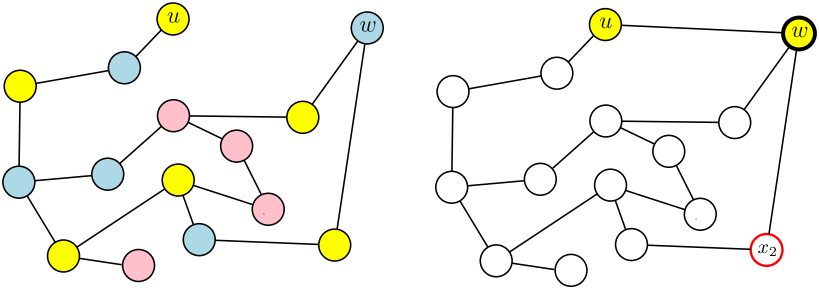

(G, σ) $\quad$ G'

we want to avoid a disagreement at x_2

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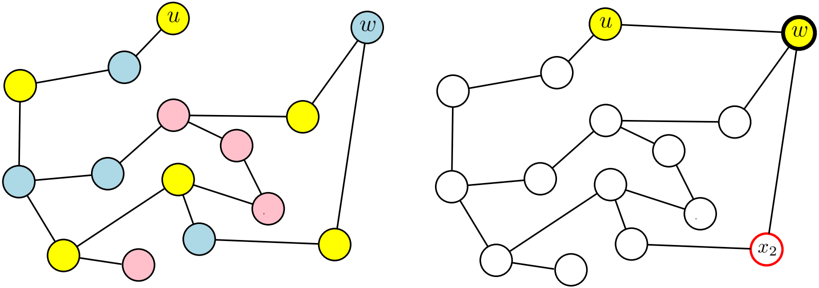

(G, σ) G'

the probability of disagreement is minimised by **coupling maximally** the marginals of μ and μ' on x_2

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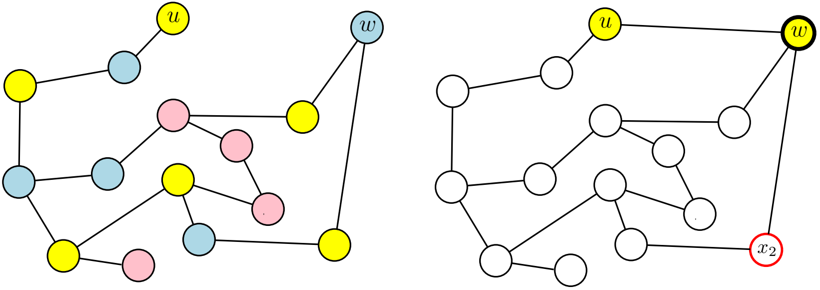

(G, σ) G'

maximal coupling

$$\Pr[\tau(x_2) = \texttt{blue}] = \max\left\{0, 1 - \frac{\mu'_{x_2}(\sigma(x_2) \mid \tau(\{u,w\}))}{\mu_{x_2}(\sigma(x_2) \mid \sigma(\{u,w\}))}\right\}.$$

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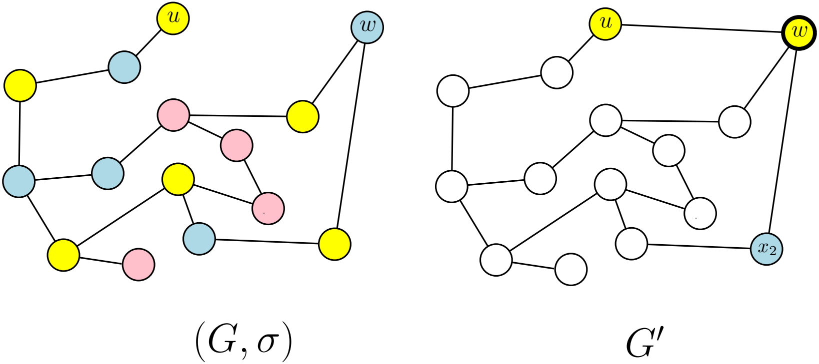

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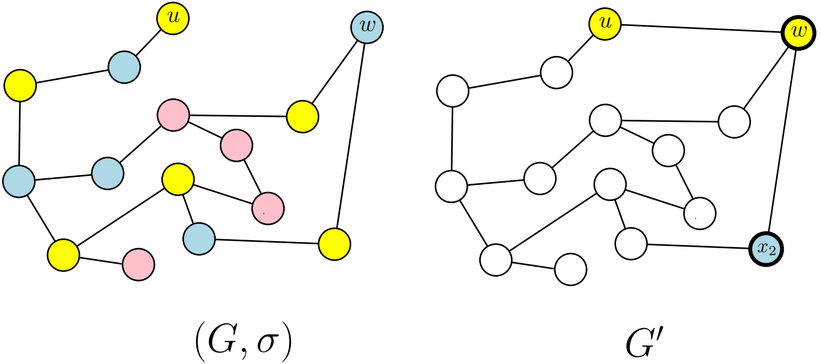
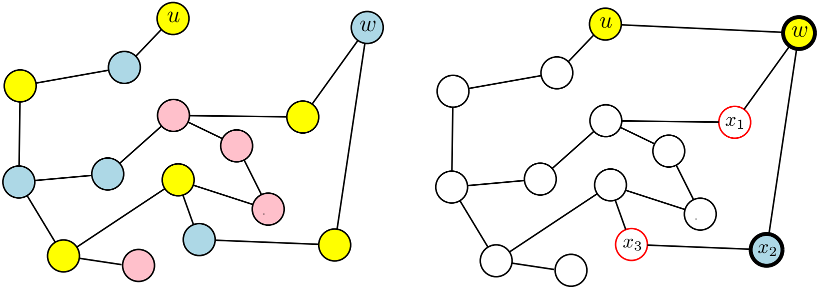


the disagreement set now is $\{w, x_2\}$

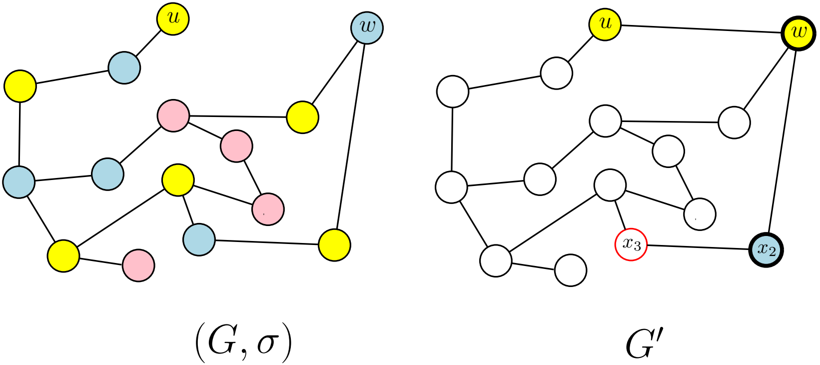
Ideal Solution



(G, σ) $\quad\quad$ G'

look for vertices z next to the disagreements such that
$\sigma(z) \in \{\texttt{blue}, \texttt{yellow}\}$

Ideal Solution



choose x_3 and repeat as before . . .

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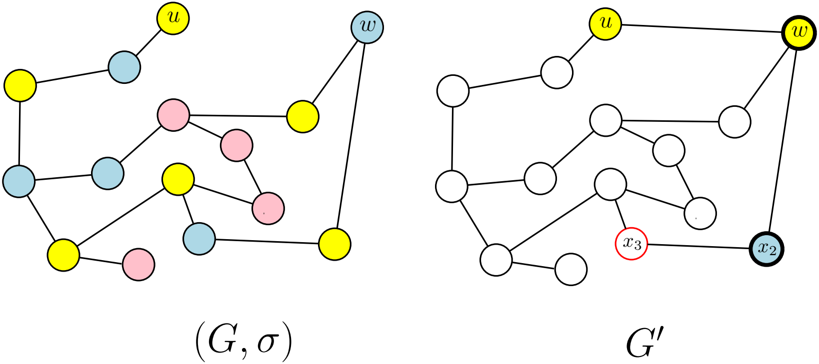

$$\Pr[\tau(x_3) = \texttt{yellow}] = \max\left\{0, 1 - \frac{\mu'_{x_3}(\sigma(x_3) \mid \tau(\{u,w,x_2\}))}{\mu_{x_3}(\sigma(x_3) \mid \sigma(\{u,w,x_2\}))}\right\}.$$

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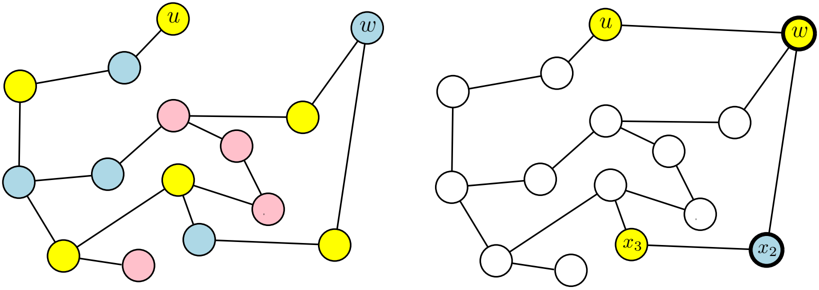

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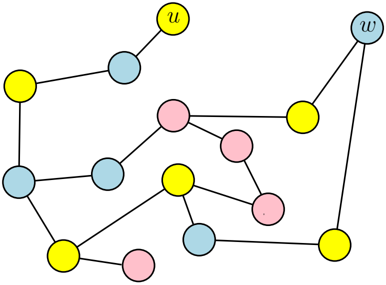

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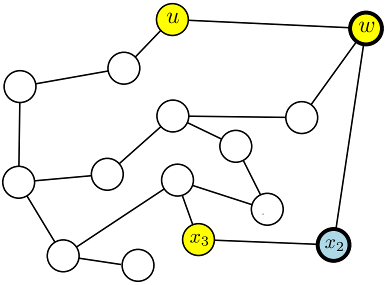

G'

repeat in the same way for the rest of the vertices

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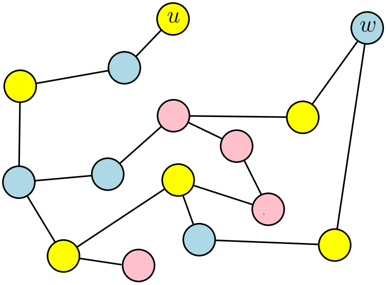

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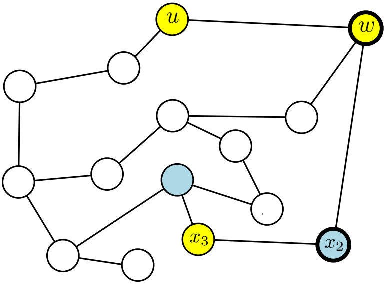

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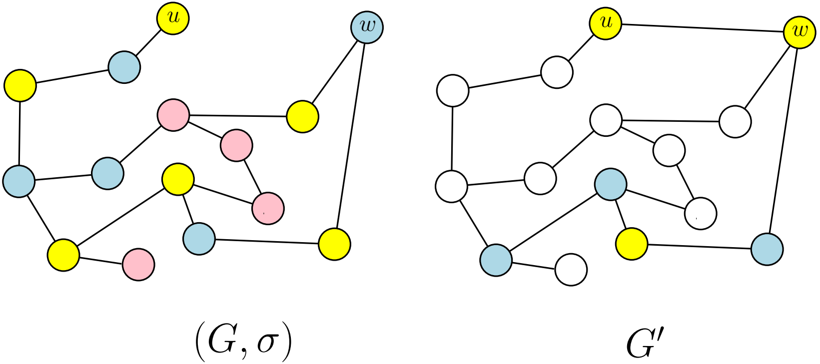

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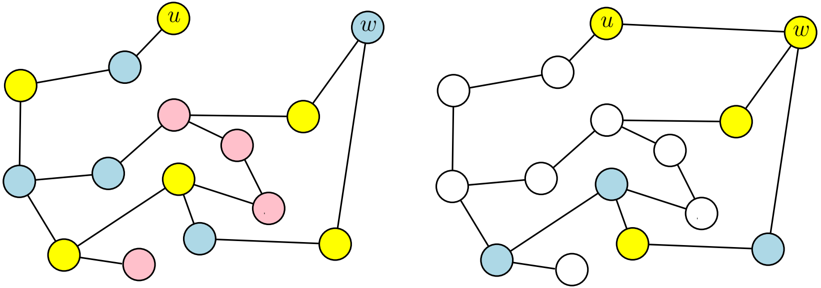

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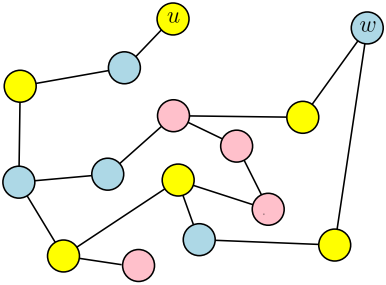

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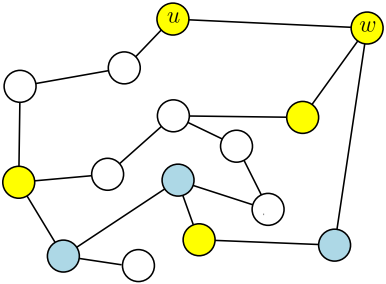

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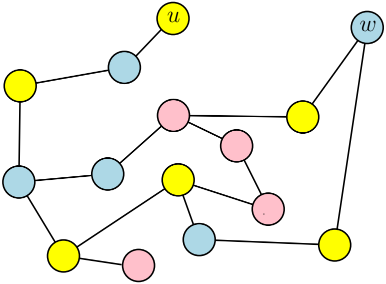

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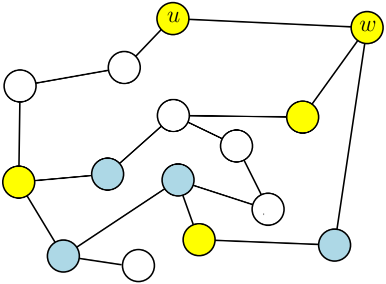

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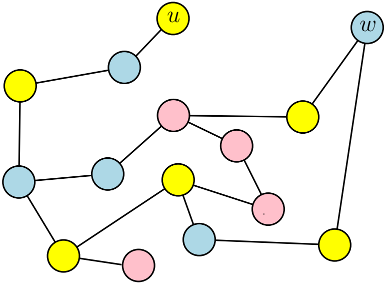

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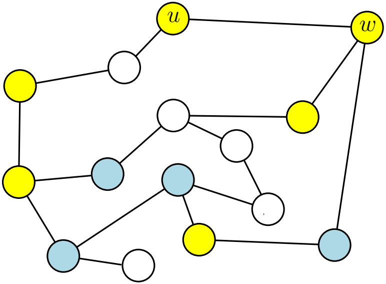

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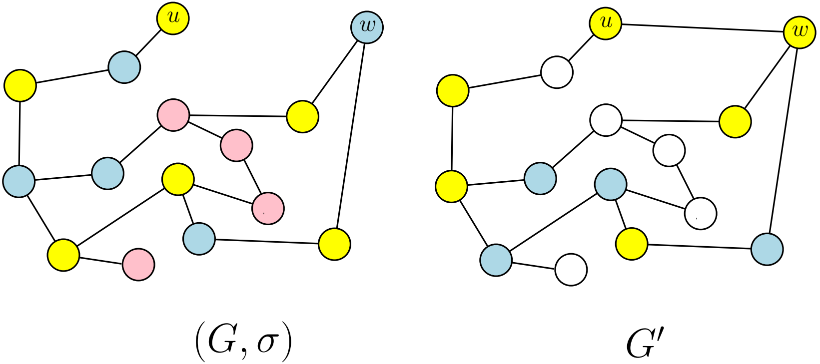
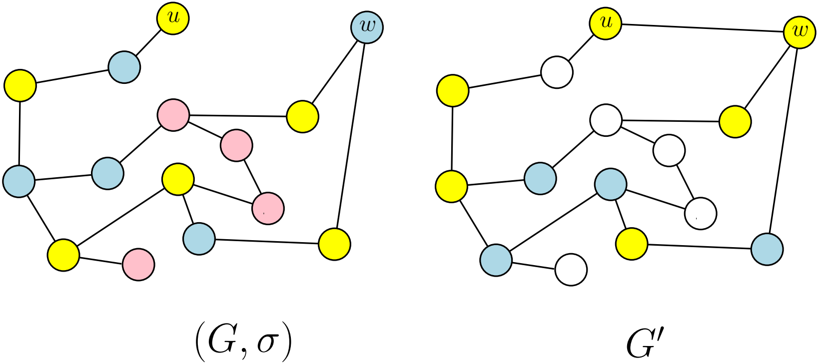


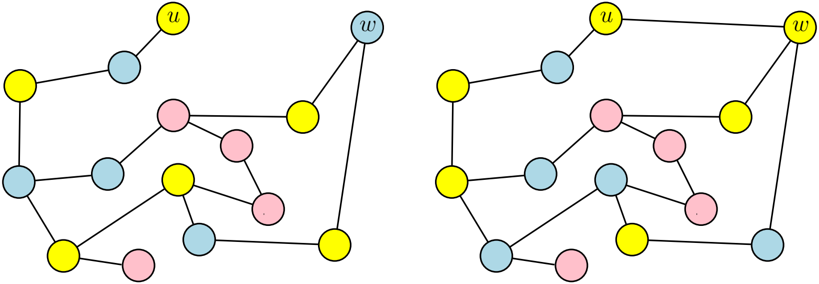
disagreement cannot propagate any more

Ideal Solution



the remaining vertices keep their assignments

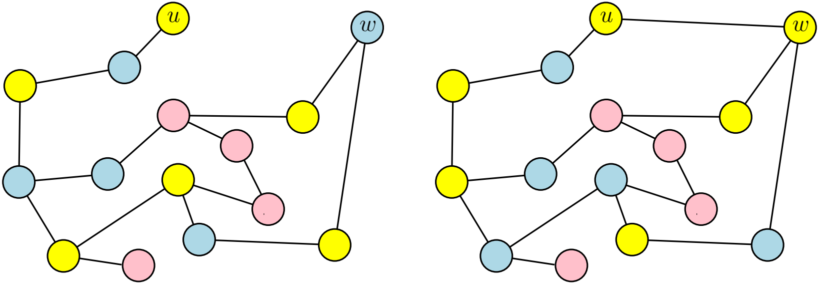
Ideal Solution



(G, σ) (G', τ)

the remaining vertices keep the initial assignments.

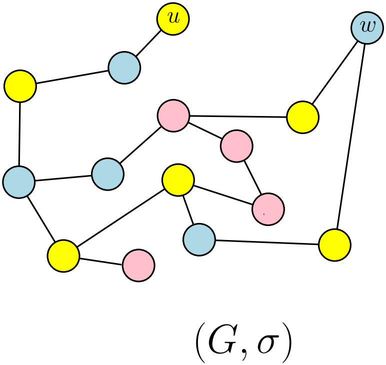
Ideal Solution

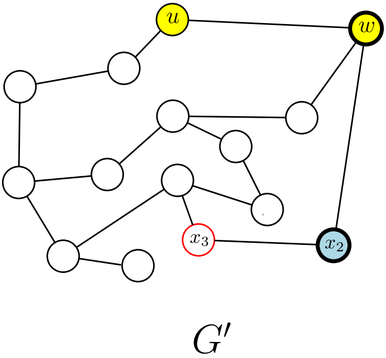


(G, σ) (G', τ)

the approach generates a **perfect** sample from μ'

Ideal Solution

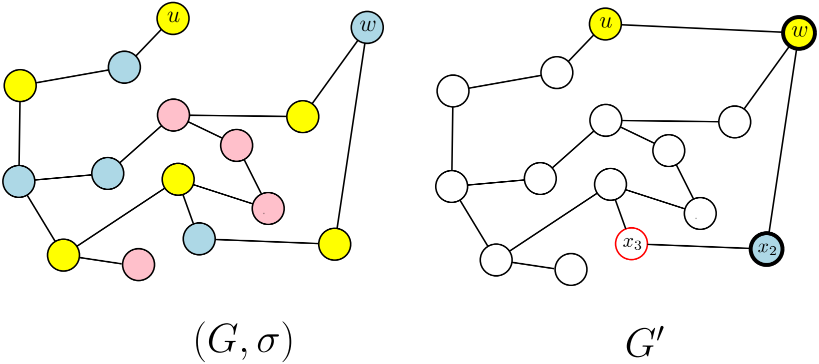




The catch ...

$$\Pr[\tau(x_3) = \texttt{yellow}] = \max\left\{0, 1 - \frac{\mu'_{x_3}(\sigma(x_3) \mid \tau(\{u,w,x_2\}))}{\mu_{x_3}(\sigma(x_3) \mid \sigma(\{u,w,x_2\}))}\right\}.$$

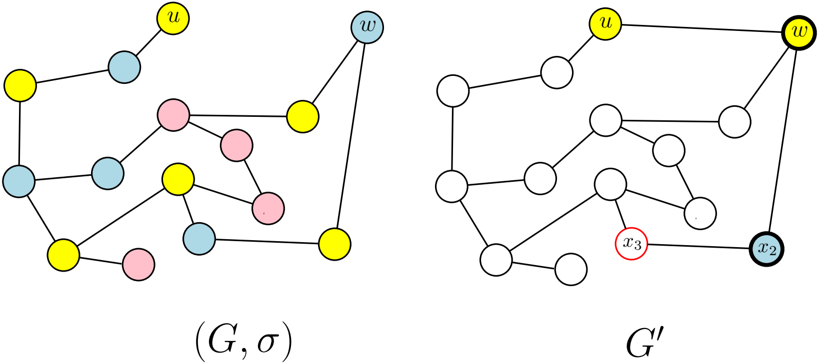
Ideal Solution



The catch ...

we need to compute $\mu'_{x_3}(\sigma(x_3) \mid \sigma(\{u, w, x_2\}))$ efficiently

Ideal Solution

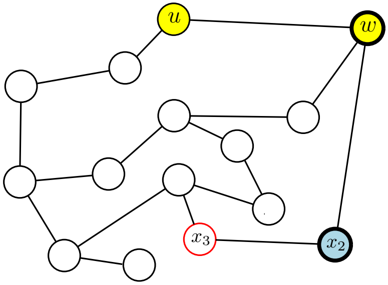


The idea . . .

replace the Gibbs marginals with “good” approximations

Some Intuition

Some Intuition

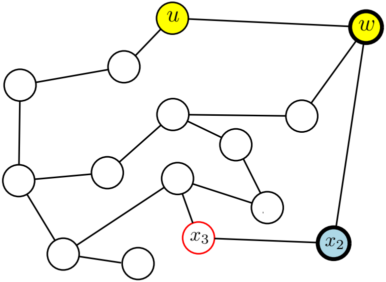


G'

Desideratum ...

compute efficiently the Gibbs marginal $\mu'_{x_3}(\sigma(x_3) \mid \sigma(\{u, w, x_2\}))$

Some Intuition

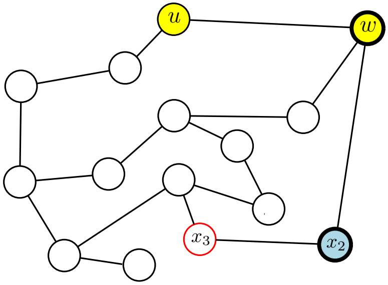


G'

Remark

the marginal $\mu'_{x_3}(\sigma(x_3) \mid \sigma(\{u, w, x_2\}))$ is a "complicated object"

Some Intuition

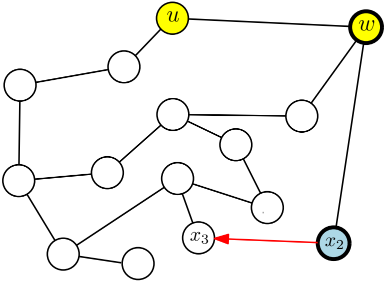


G'

Observation ...

influences form vertices with configuration make the Gibbs marginal at x_3 complicated

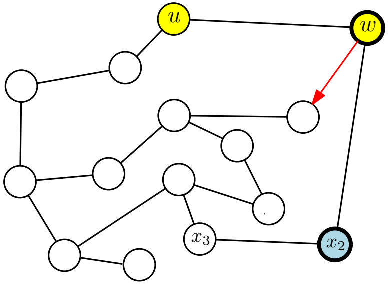
Some Intuition



G'

influence from the configuration at x_2

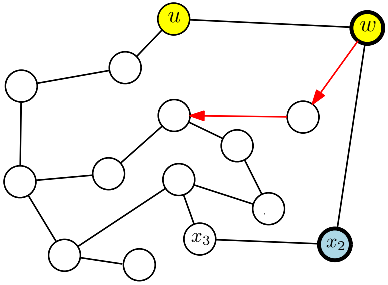
Some Intuition



G'

influence from the configuration at w

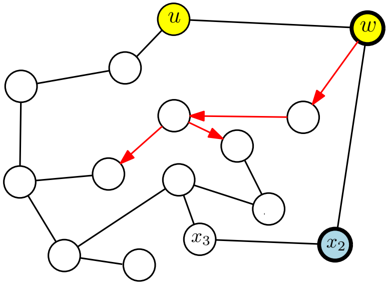
Some Intuition



G'

influence from the configuration at w

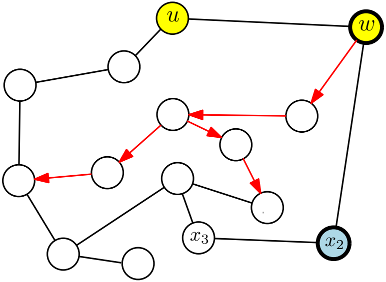
Some Intuition



G'

influence from the configuration at w

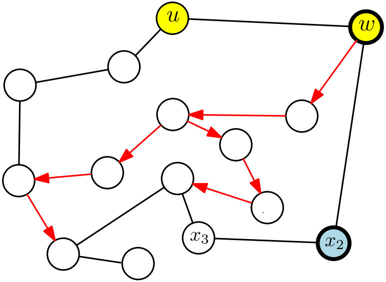
Some Intuition



G'

influence from the configuration at w

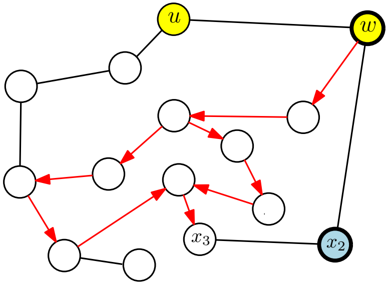
Some Intuition



G'

influence from the configuration at w

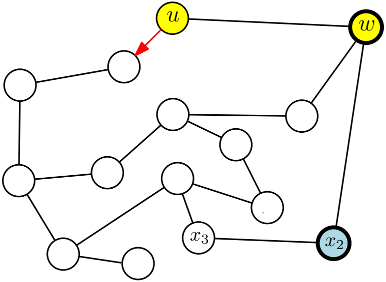
Some Intuition



G'

influence from the configuration at w

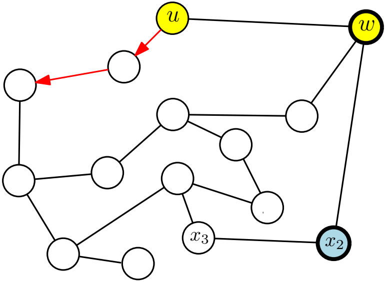
Some Intuition



G'

influence from the configuration at u

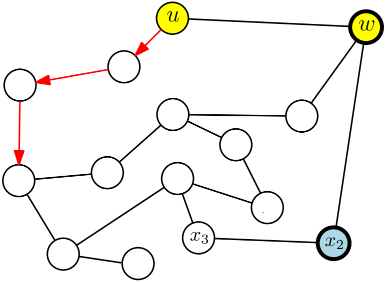
Some Intuition



G'

influence from the configuration at u

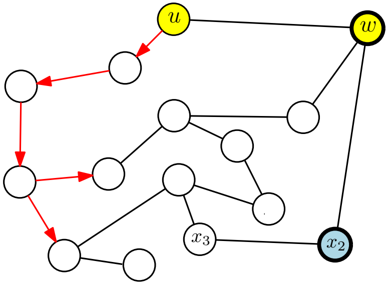
Some Intuition



G'

influence from the configuration at u

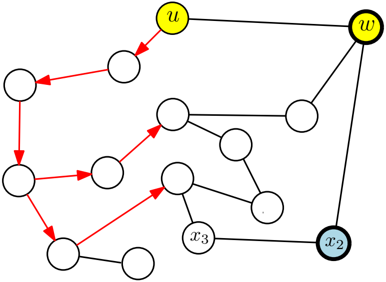
Some Intuition



$$G'$$

influence from the configuration at u

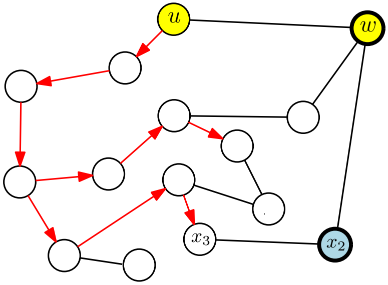
Some Intuition



G'

influence from the configuration at u

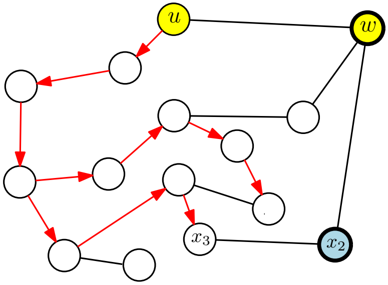
Some Intuition



G'

influence from the configuration at u

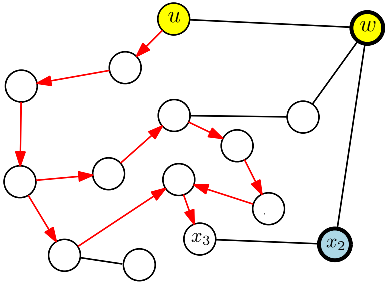
Some Intuition



G'

influence from the configuration at u

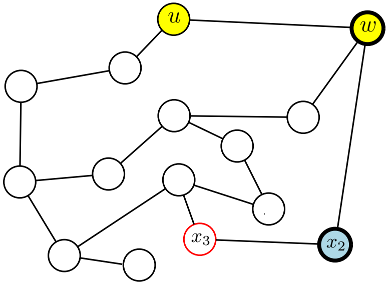
Some Intuition



G'

influence from the configuration at u

Some Intuition

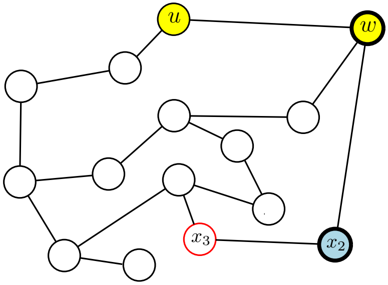


G'

However . . .

in most cases all but one vertex are far away (**girth**)

Some Intuition

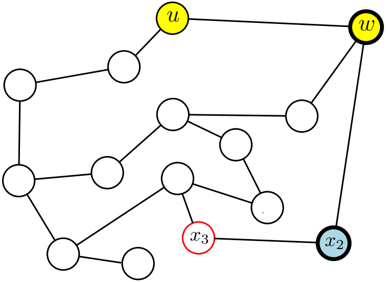


G'

Choosing the appropriate parameters . . .

the influences from distance are very weak & in most cases only one vertex influences the marginal

Some Intuition

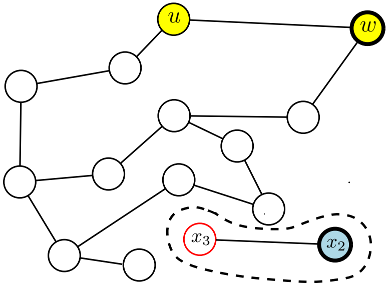


G'

Compute marginal but ...

ignore the influence on x_3 from u and w

Some Intuition

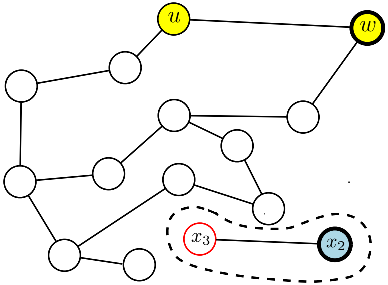


G'

Effectively

use the marginal of x_3 on the graph within the dashed curve

Some Intuition

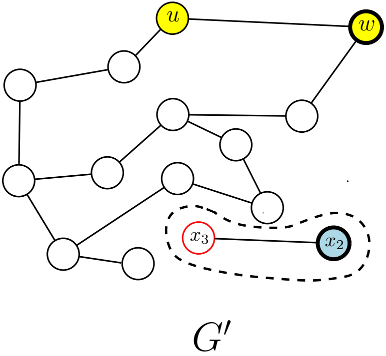


G'

Remarks

the "simplified" marginal on x_3 is trivial & is computed fast

Some Intuition

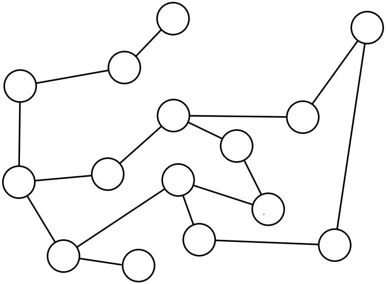


Remarks

our marginal is also called **broadcasting probability**

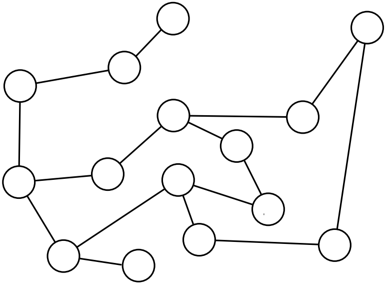
To sum up . . .

To sum up . . .

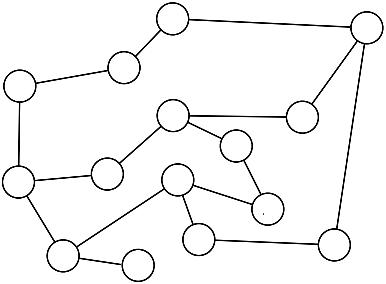


G

To sum up . . .

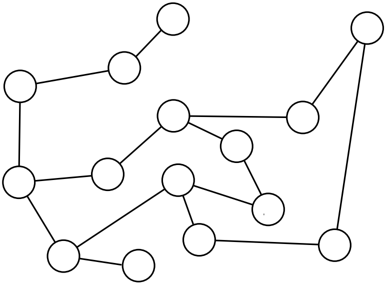


G

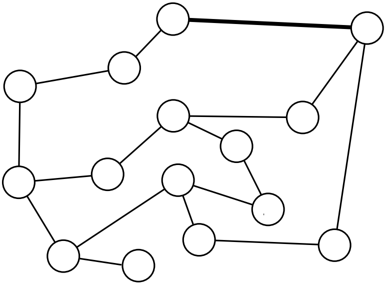


G'

To sum up . . .

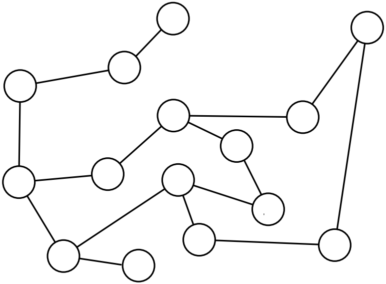


G

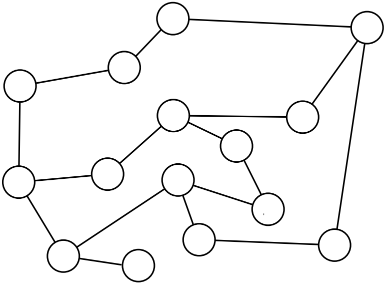


G'

To sum up . . .

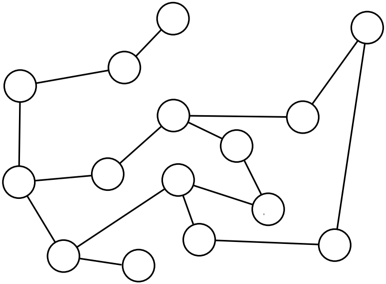


G

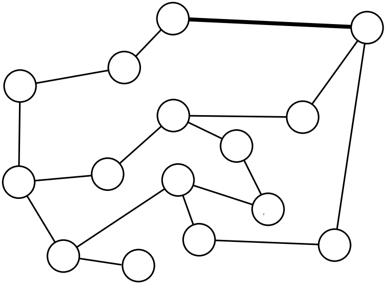


G'

To sum up . . .

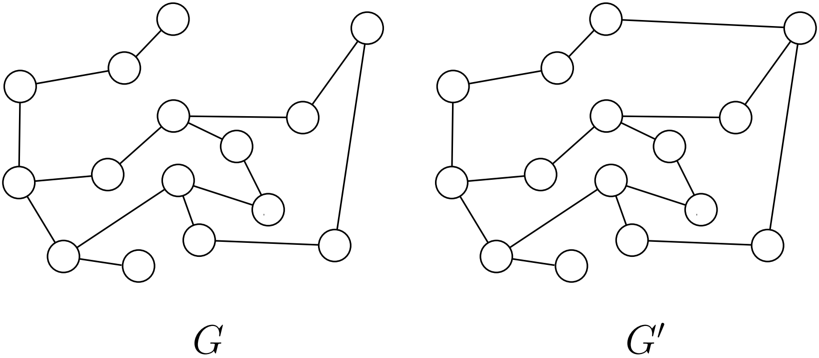


G

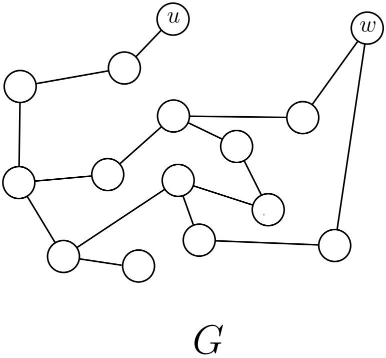


G'

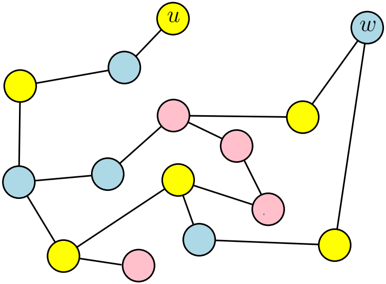
To sum up . . .



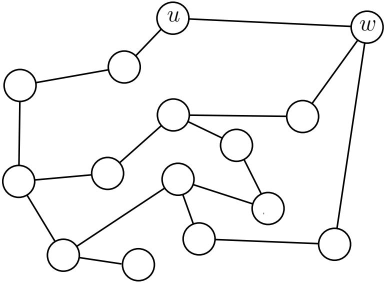
To sum up . . .



To sum up . . .

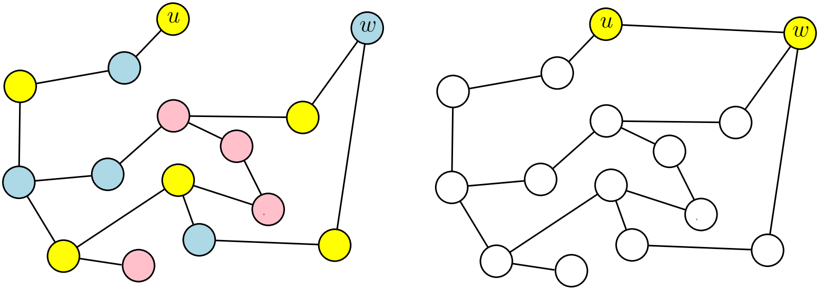


(G, σ)



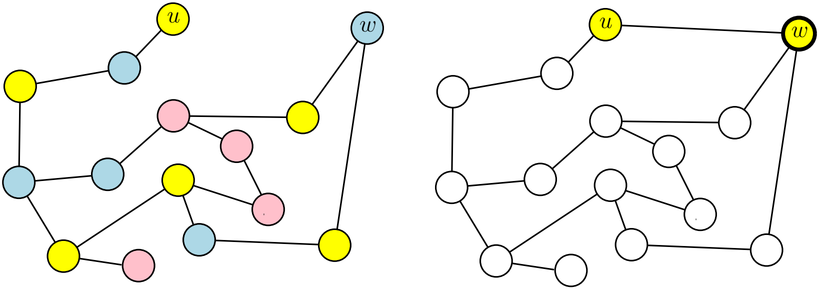
G'

To sum up . . .



(G, σ) G'

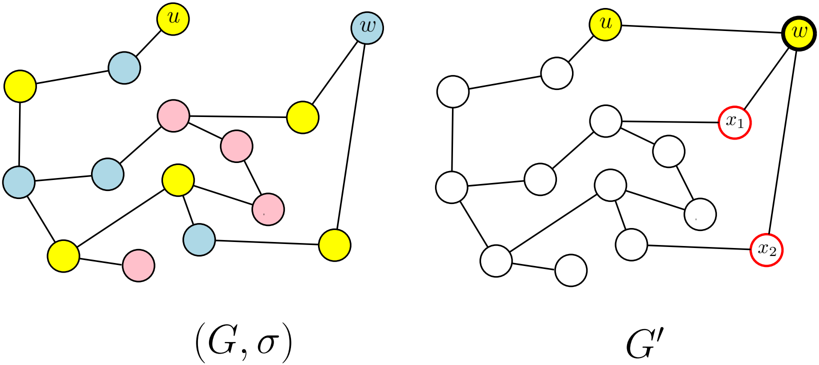
To sum up . . .



(G, σ) $\qquad$ G'

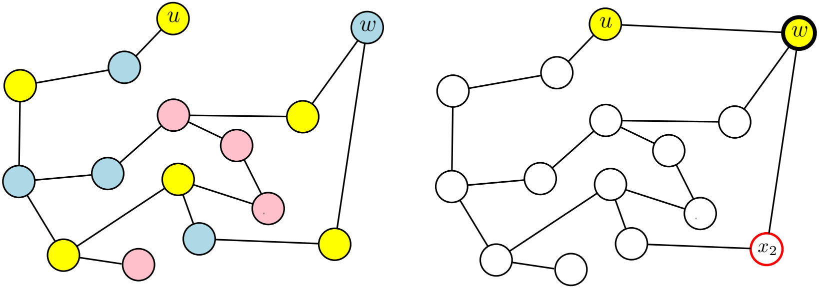
vertex w is a **disagreement** with spins $\mathcal{D} = \{\texttt{blue}, \texttt{yellow}\}$

To sum up . . .



look for z, neighbour of w with $\sigma(z) \in \{\texttt{blue}, \texttt{yellow}\}$

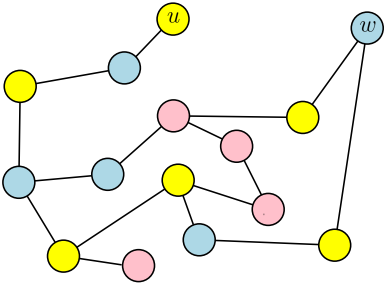
To sum up . . .



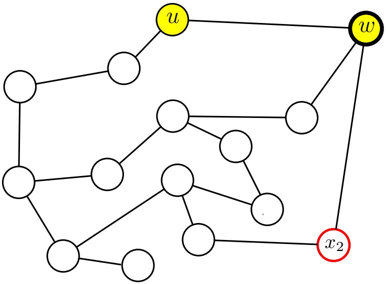
(G, σ) G'

pick x_2 and decide $\tau(x_2)$ such that $\tau(x_2) \in \{\texttt{blue}, \texttt{yellow}\}$

To sum up . . .



(G, σ)

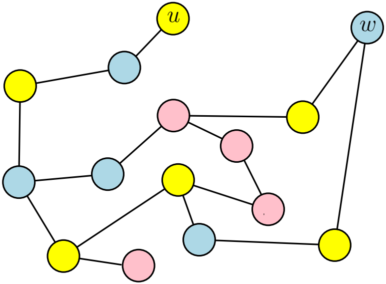


G'

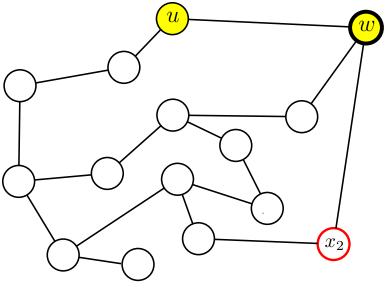
maximal coupling of broadcasting probabilities

$$\Pr[\tau(x_2) = \texttt{blue}] = \max\left\{0, 1 - \frac{\mathfrak{m}_{x_2}(\sigma(x_2) \mid \tau(w))}{\mathfrak{m}_{x_2}(\sigma(x_2) \mid \sigma(w))}\right\}$$

To sum up . . .



(G, σ)

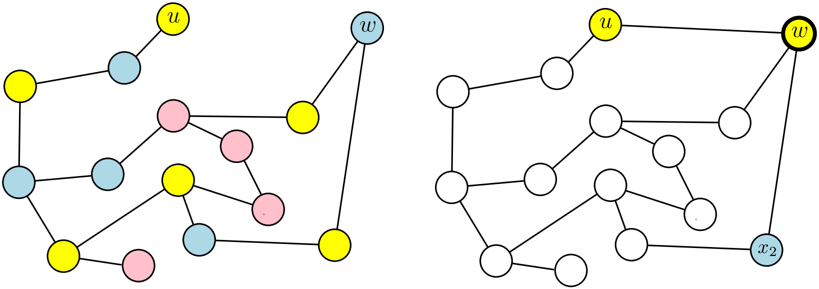


G'

maximal coupling of broadcasting probabilities

$$\Pr[\tau(x_2) = \texttt{blue}] = \max\left\{0, 1 - \frac{\mathfrak{m}_{x_2}(\sigma(x_2) \mid \tau(w))}{\mathfrak{m}_{x_2}(\sigma(x_2) \mid \sigma(w))}\right\}$$

To sum up . . .

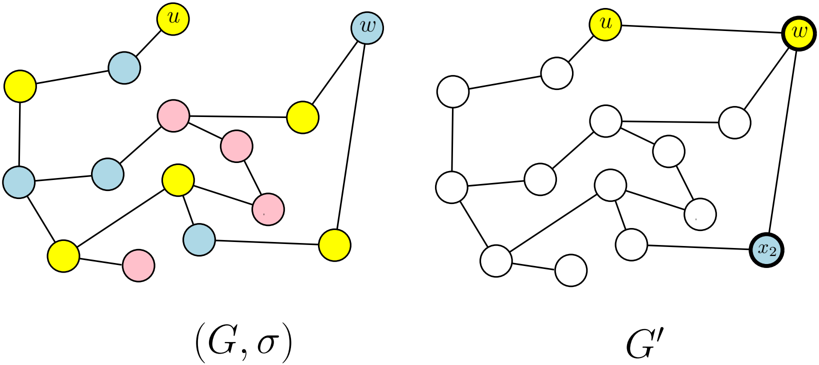


(G, σ) $\qquad$ G'

maximal coupling of broadcasting probabilities

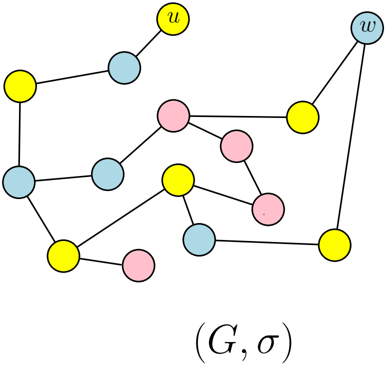
$$\Pr[\tau(x_2) = \texttt{blue}] = \max\left\{0, 1 - \frac{\mathfrak{m}_{x_2}(\sigma(x_2) \mid \tau(w))}{\mathfrak{m}_{x_2}(\sigma(x_2) \mid \sigma(w))}\right\}$$

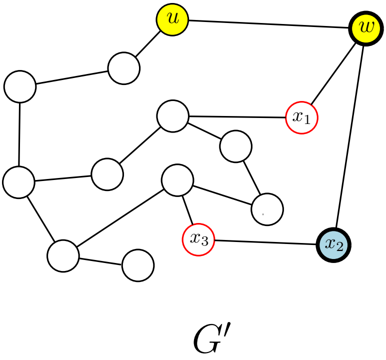
To sum up . . .



the disagreement set is $\{w, x_2\}$

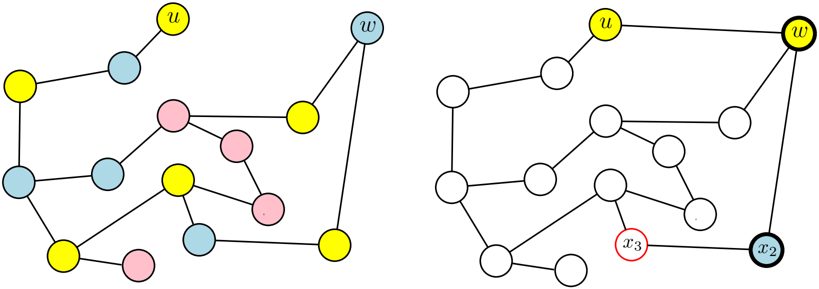
To sum up . . .





look for vertices z next to the disagreements such that $\sigma(z) \in \{\texttt{blue}, \texttt{yellow}\}$

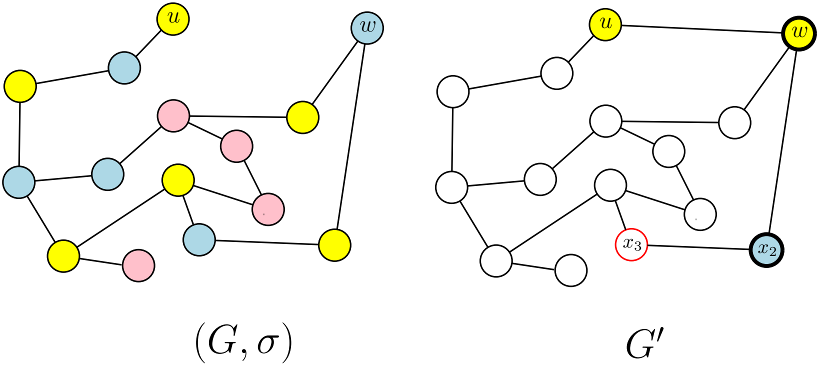
To sum up . . .



(G, σ) $\qquad$ G'

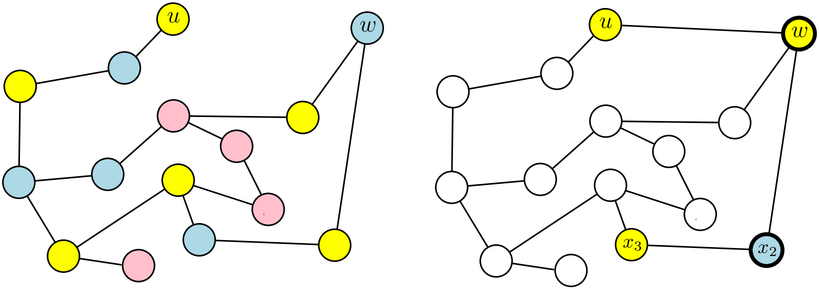
choose x_3 and repeat as before . . .

To sum up . . .



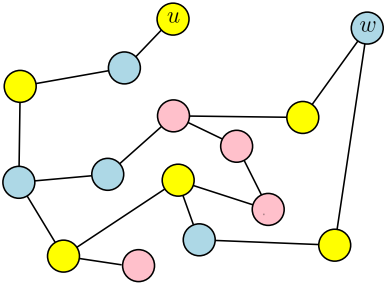
$$\Pr[\tau(x_3) = \texttt{yellow}] = \max\left\{0, 1 - \frac{\mathfrak{m}_{x_3}(\sigma(x_3) \mid \tau(x_2))}{\mathfrak{m}_{x_3}(\sigma(x_3) \mid \sigma(x_2))}\right\}.$$

To sum up . . .

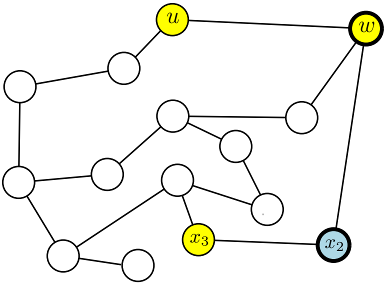


(G, σ) G'

To sum up . . .



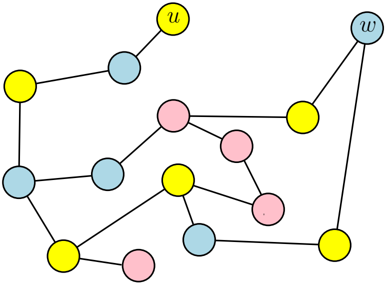
(G, σ)



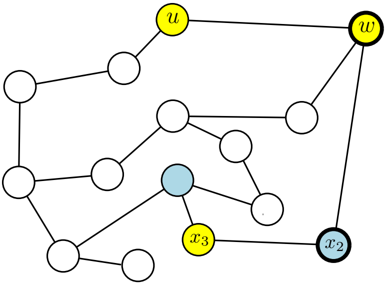
G'

repeat in the same way for the rest of the vertices

To sum up . . .



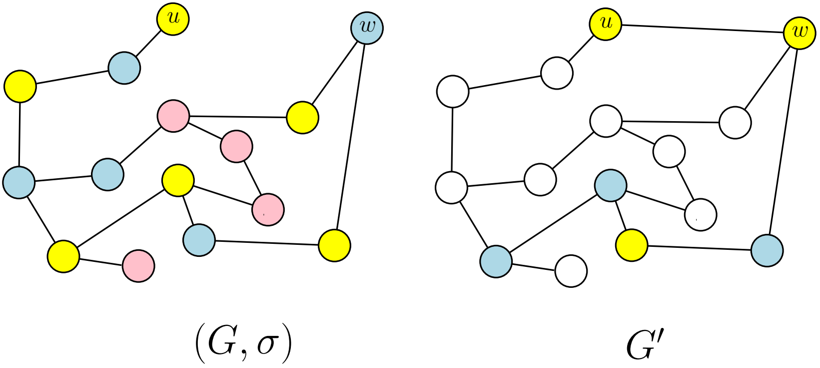
(G, σ)



G'

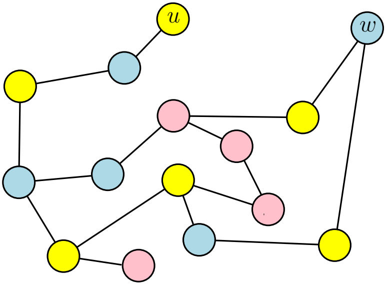
repeat in the same way for the rest of the vertices

To sum up . . .

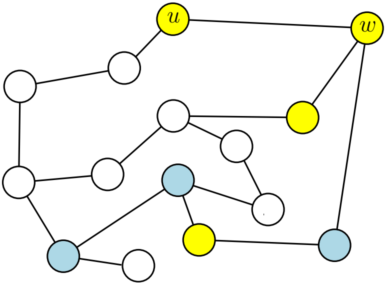


repeat in the same way for the rest of the vertices

To sum up . . .



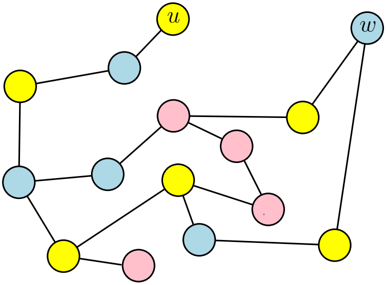
(G, σ)



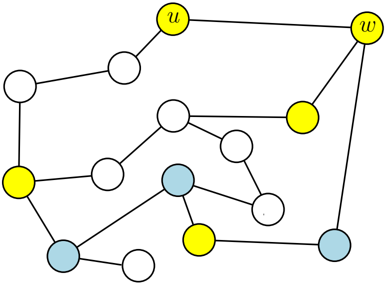
G'

repeat in the same way for the rest of the vertices

To sum up . . .



(G, σ)



G'

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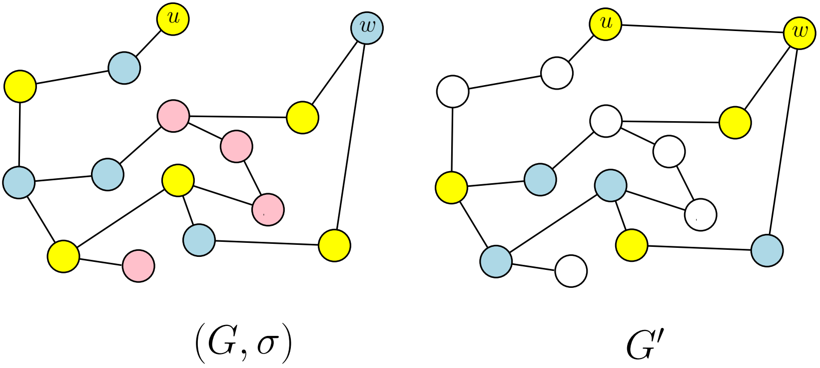

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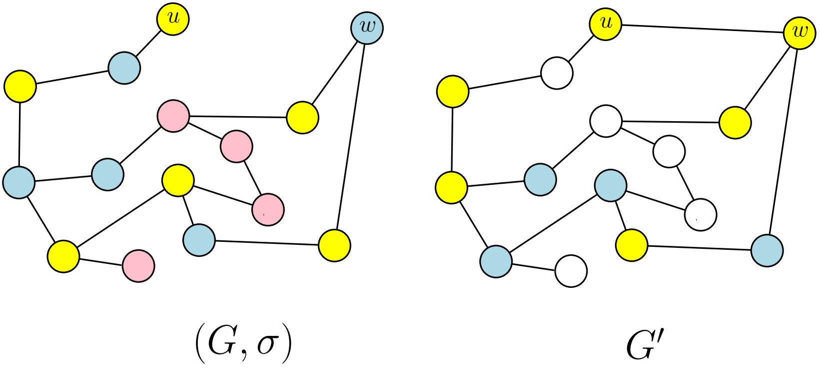

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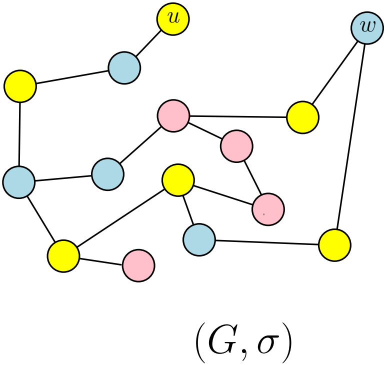
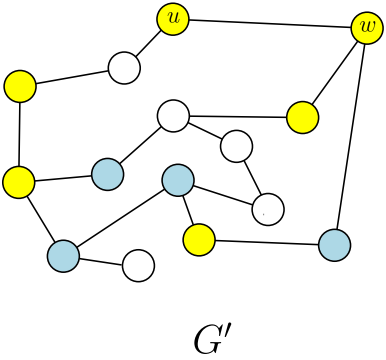
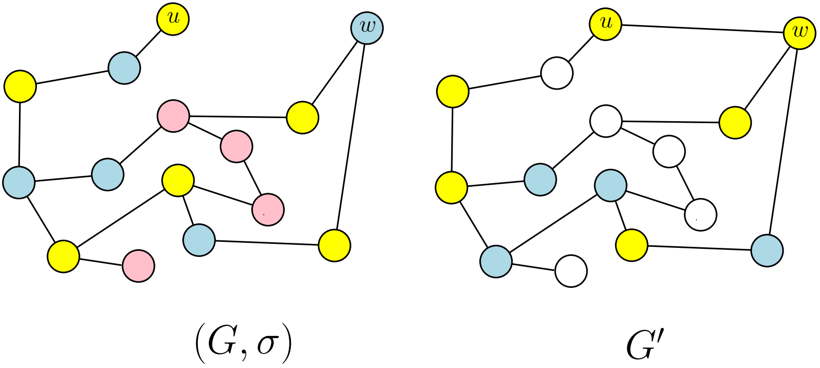




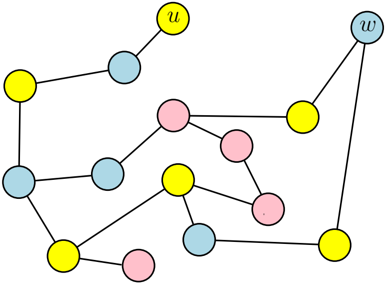
the disagreements cannot propagate any more

To sum up . . .



the remaining vertices keep the same assignment

To sum up . . .



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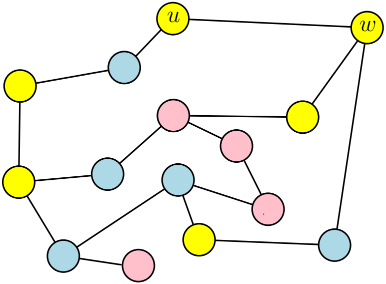
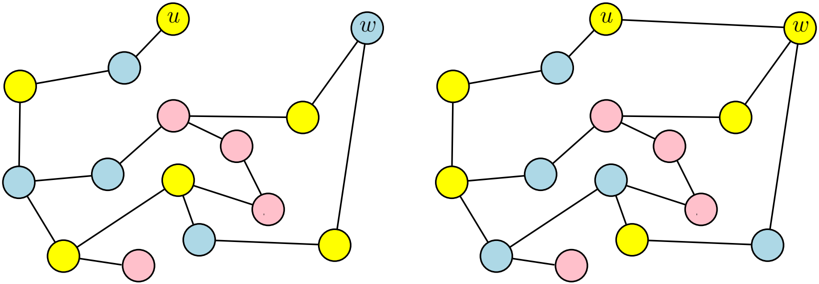


(G', τ)

the remaining vertices keep their assignments.

To sum up . . .

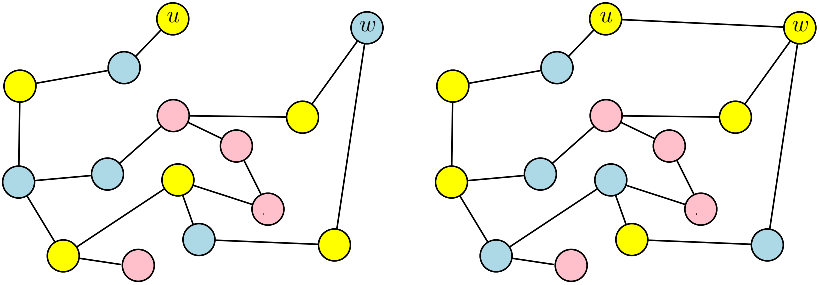


(G, σ) (G', τ)

The catch . . .

the process is not allowed to “self interact”

To sum up . . .

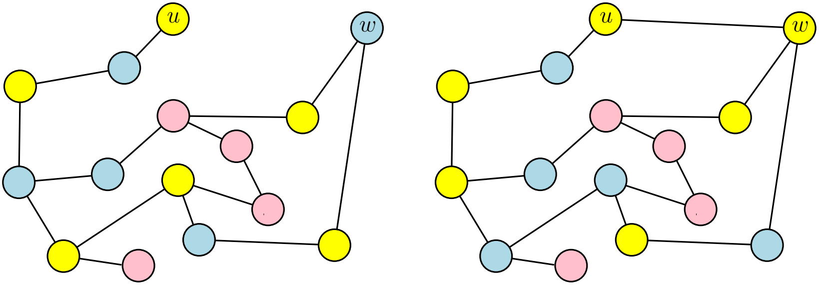


(G, σ) (G', τ)

Self interaction

- update neighbours of u
- the vertices whose assignment change induce a cycle

To sum up . . .



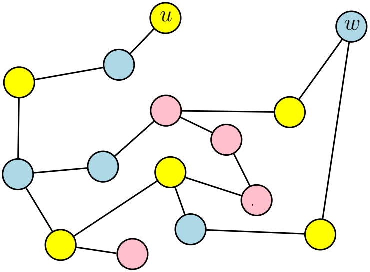
(G, σ) (G', τ)

Failure

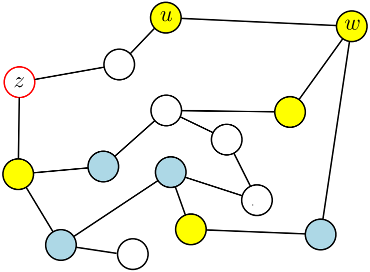
failure is when the process self-interacts.

Example of failure

Example of failure

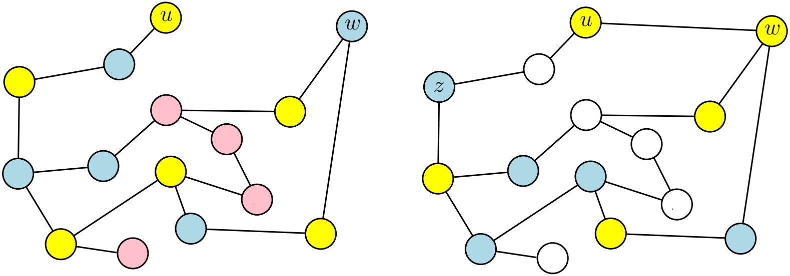


(G, σ)



G'

Example of failure



(G, σ) G'

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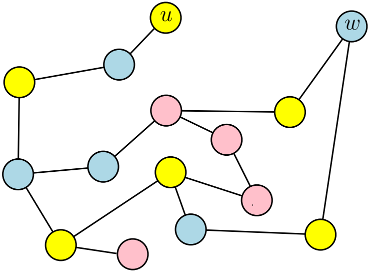

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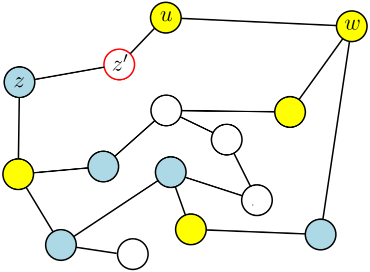
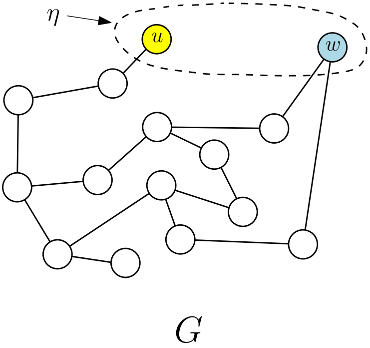
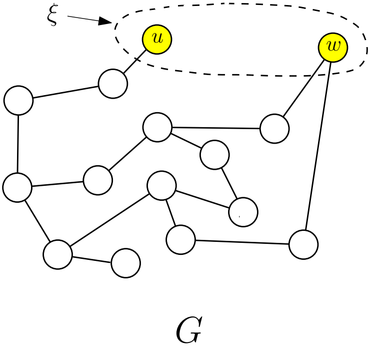


G'

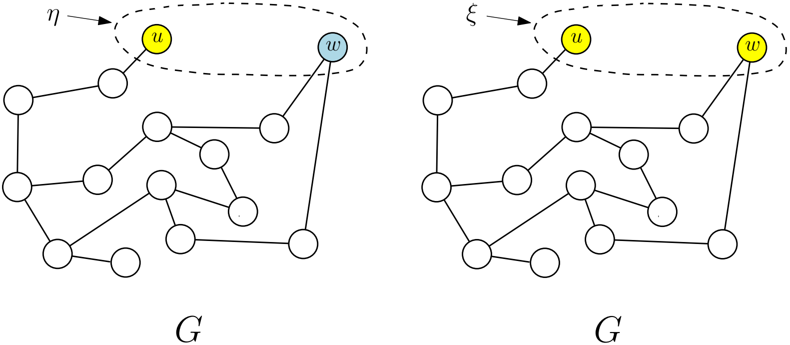
Failure Vs Approximation

Failure Vs Approximation





Failure Vs Approximation

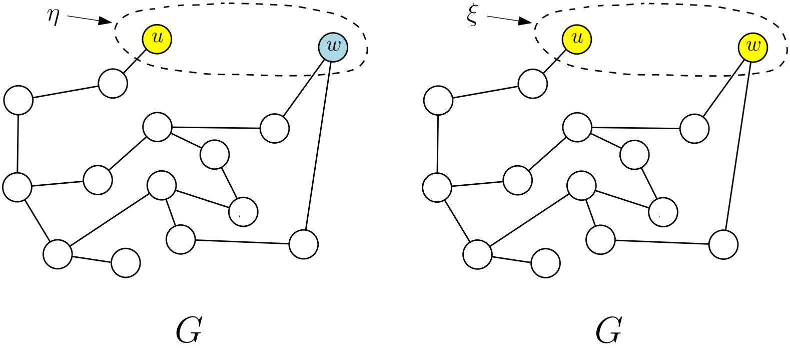


Accuracy

we have `Update` on input $\boldsymbol{\sigma}$ distributed as in $\mu(\cdot \mid \eta)$

- ν is the distribution of the output of `Update`
- compare ν with $\mu(\cdot \mid \xi)$

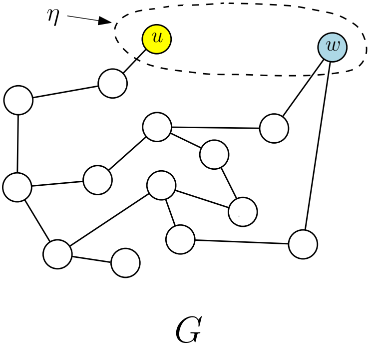
Failure Vs Approximation



Accuracy

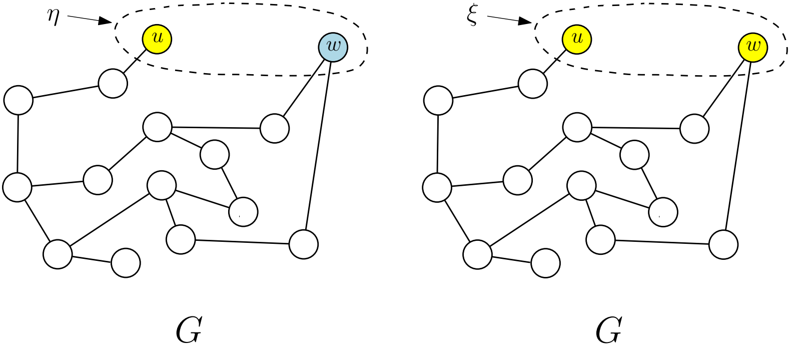
$$||\nu - \mu(\cdot \mid \xi)||_{tv} = ???$$

Failure Vs Approximation



we handle the update as a (probabilistic) map

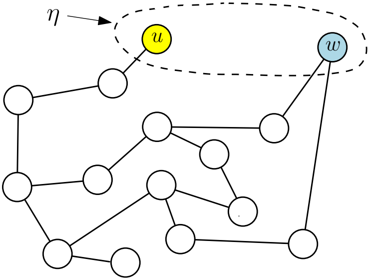
Failure Vs Approximation



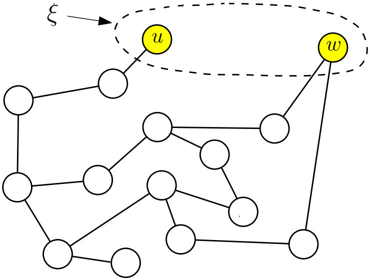
For configurations σ, τ

$\mathrm{P}_{\eta,\xi}(\sigma, \tau) :=$ Probability that update generates τ given input σ

Failure Vs Approximation



G

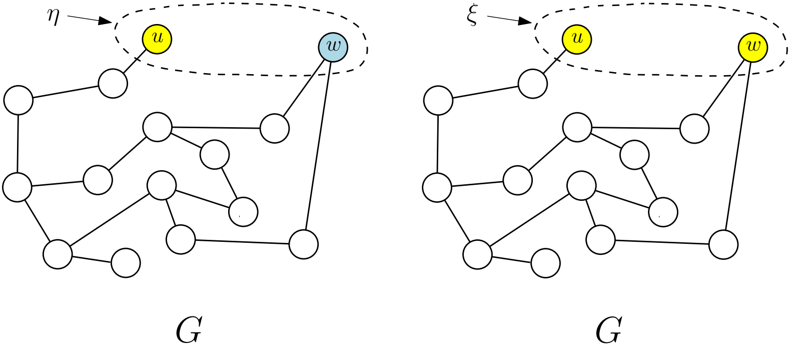


G

Reverse mapping

we can define the reverse mapping (right to left)

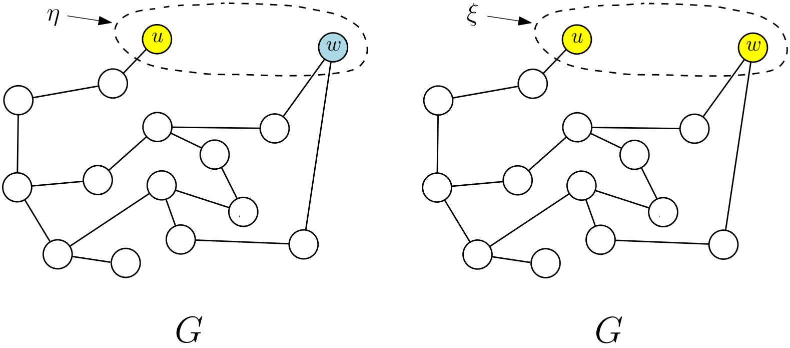
Failure Vs Approximation



For configurations σ, τ

$\mathrm{P}_{\xi,\eta}(\tau, \sigma) :=$ Probability that reverse update generates σ on input τ

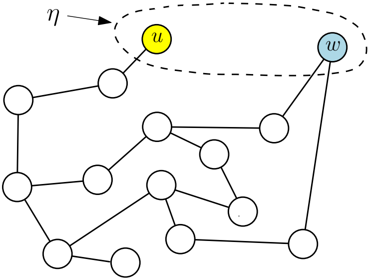
Failure Vs Approximation



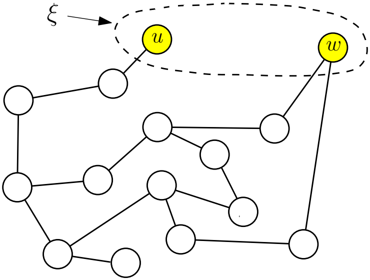
Transition probabilities

$P_{\eta,\xi}(\cdot,\cdot) \Rightarrow$ for the mapping from left to right
$P_{\xi,\eta}(\cdot,\cdot) \Rightarrow$ for the mapping from right to left

Failure Vs Approximation



G



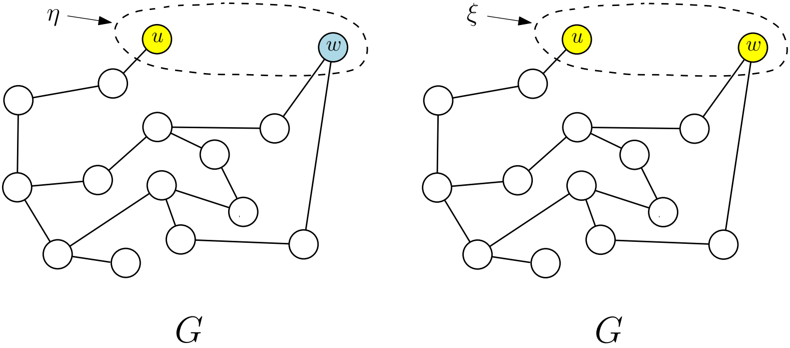
G

Detailed Balance Property

For any σ, τ we have

$$\mu(\sigma)\mathrm{P}_{\eta,\xi}(\sigma,\tau) = \mu(\tau)\mathrm{P}_{\xi,\eta}(\tau,\sigma)$$

Failure Vs Approximation



Using detailed balance we get ...

$$||\nu - \mu(\cdot \mid \xi)|| \approx \frac{1}{2}(\Pr[\texttt{Update Fails}] + \Pr[\texttt{Reverse Fails}])$$

Configuration for u and w

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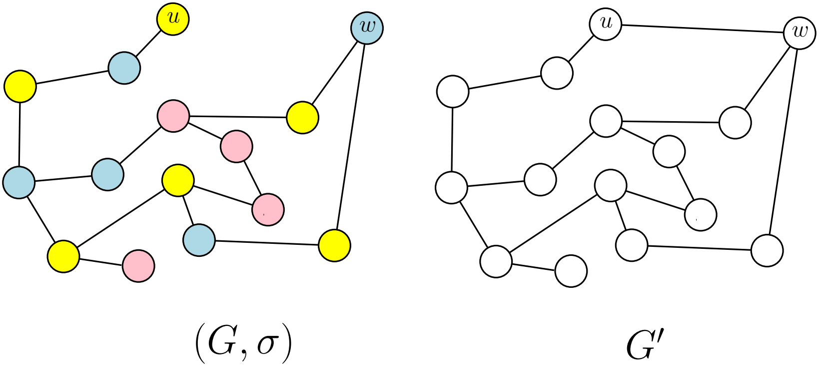

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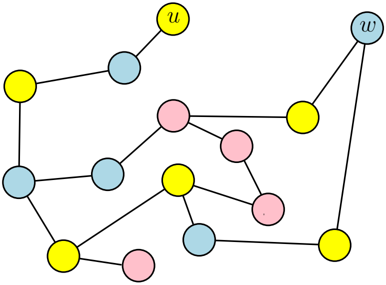

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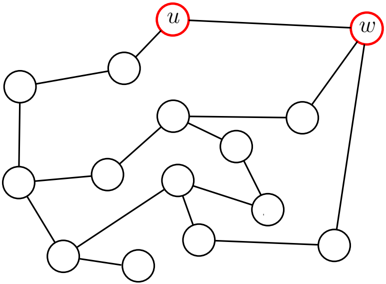

G'

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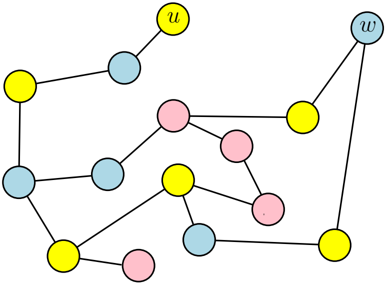

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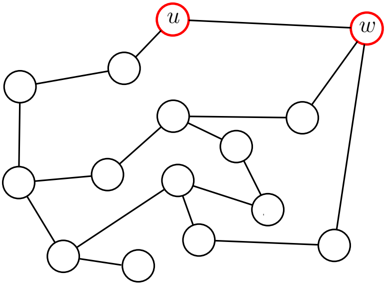
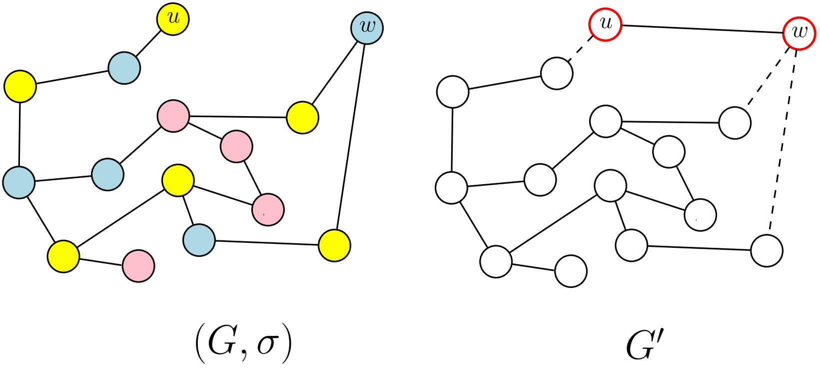


G'

Remark

The choice of $\tau(u)$ and $\tau(w)$ in G' is oblivious to σ

Configuration for u and w



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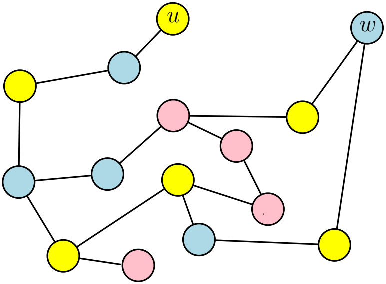

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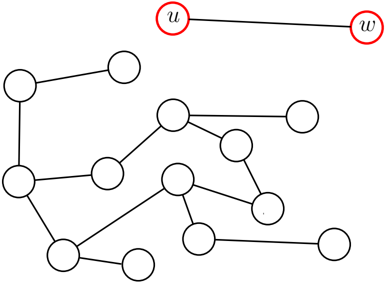

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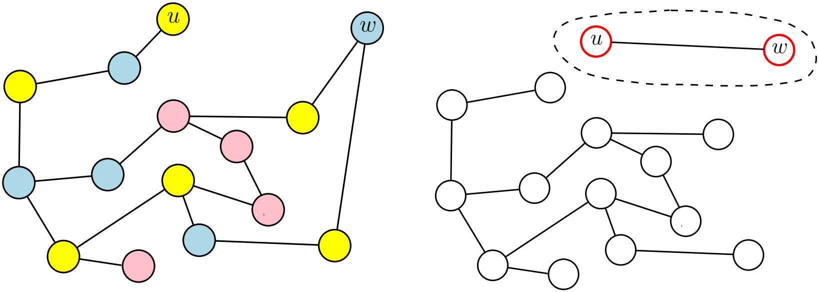
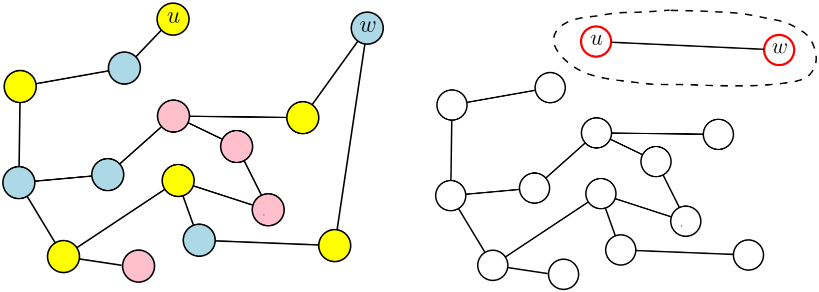


(G, σ) G'

Sample from the distribution on the graph within the dashed lines

Configuration for u and w



(G, σ) $\qquad$ G'

Remarks

- introduces an extra error
- initial disagreement maybe ≥ 1

The algorithm

The iterative algorithm

The algorithm

Input: $G = (V, E)$ $k > 0$

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$G_0, G_1, \ldots, G_r = G$

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The error for the algorithm

$\approx$ probability of failure at some iteration

The iterative algorithm

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The time complexity

the time complexity is $O(|E|^2)$

- for each iteration we compute $O(|E|)$ broadcasting marginals
- we have $|E|$ iterations

From high girth to $\boldsymbol{G}(n, m)$

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  - there are short cycles far apart from each other
- we won't discuss the challenges from the short cycles here ...

The parameters

The parameters

For which parameters of the Gibbs distribution on $G(n, m)$ do we get good approximations?

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- Gibbs uniqueness condition
- tools based on Contiguity

Analysis

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Consider $\boldsymbol{G}(n, m)$ and $\boldsymbol{\sigma}$ distributed as in μ. Consider `Update` that starts from vertex v with initial assignment $\tau(v) \neq \boldsymbol{\sigma}(v)$

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The disagreements grow **subcritically**

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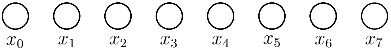
The randomness ...

- the random graph $G(n, m)$
- confinguration $\boldsymbol{\sigma}$
- random choices of `Update`

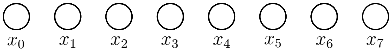
Analysis Sketch

Analysis Sketch



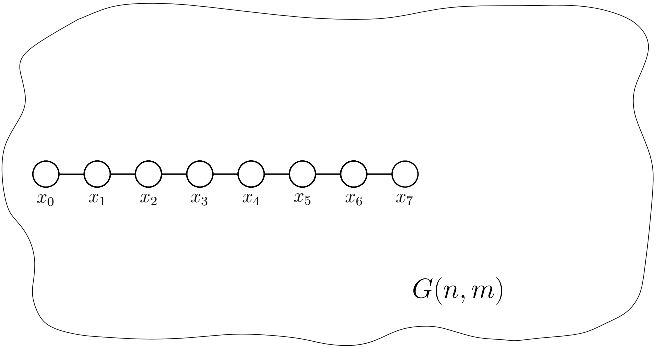
consider a permutation of vertices s.t. $x_0 = v$

Analysis Sketch



reveal the graph structure $G(n, m)$

Analysis Sketch



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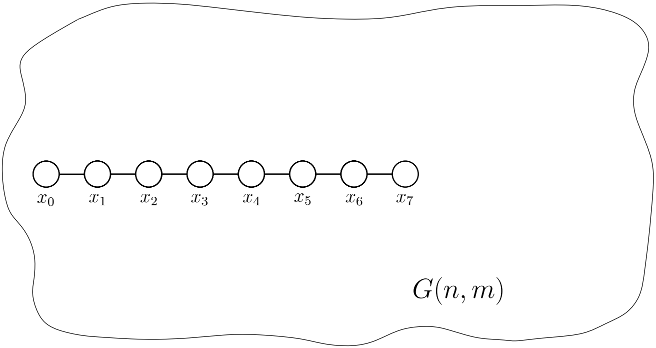
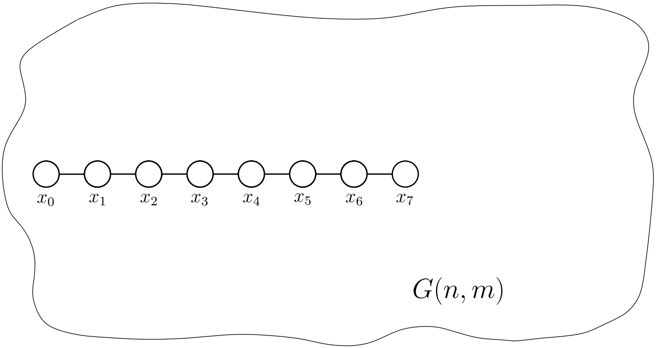
Analysis Sketch



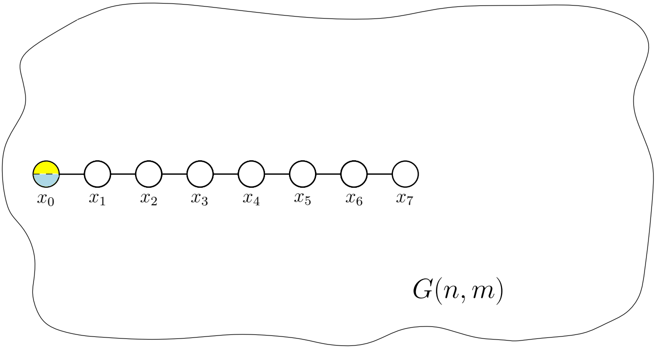
we care whether $x_0, \ldots, x_7$ forms a path in $\boldsymbol{G}(n, m)$

Analysis Sketch



we set initial disagreement at x_0

Analysis Sketch



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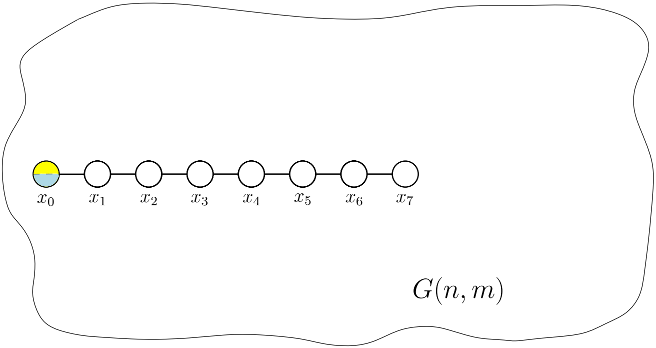
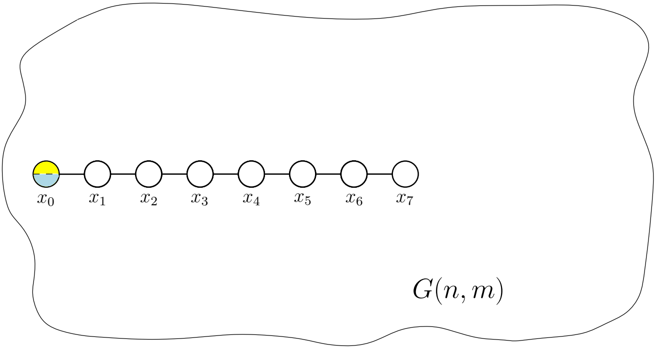


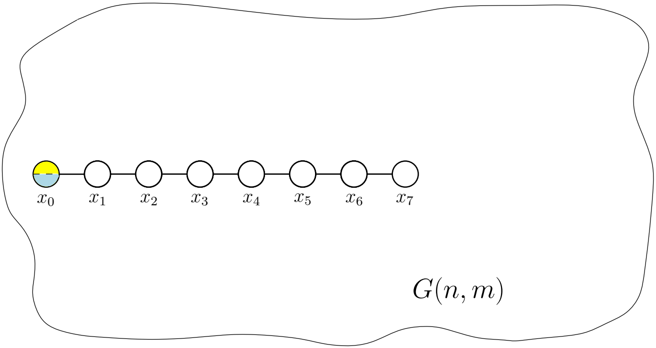
the two configurations at x_0 are from the input $\boldsymbol{\sigma}$ and he output $\boldsymbol{\tau}$ of `Update`

Analysis Sketch



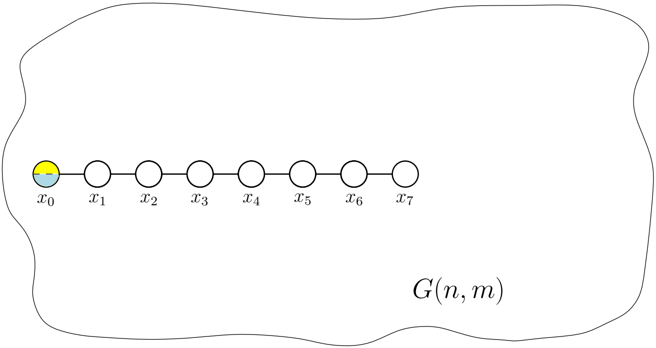
focus is on the **probability** that the disagreement propagates over the path $x_0, \ldots, x_7$

Analysis Sketch



in steps, for $x_1, x_2, \ldots$, we reveal the configurations on the vertices in the path

Analysis Sketch



Disagreement probability

the probability that the disagreement propagates one step further

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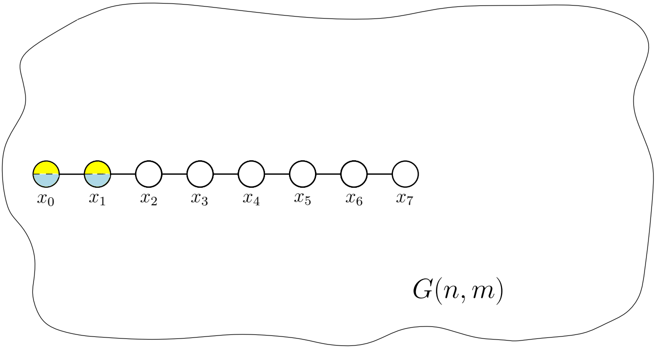

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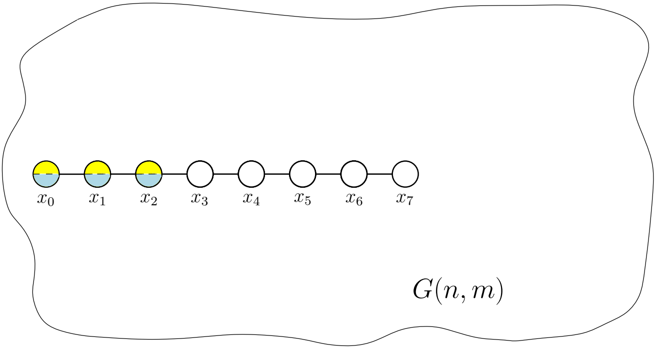

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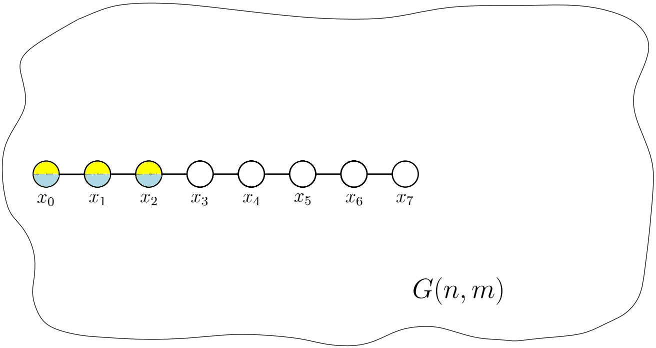

Desideratum

at each step the disagreement probability $< 1/d$

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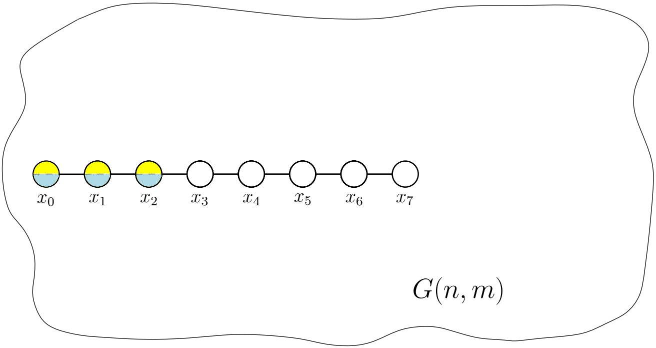
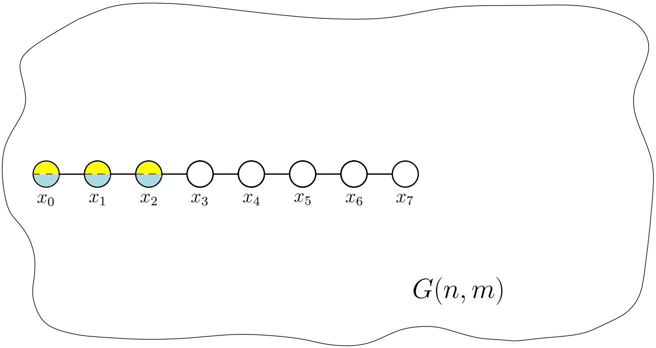


Remark I

this probability depends on μ and the random choice of `Update`

Analysis Sketch



Some magic

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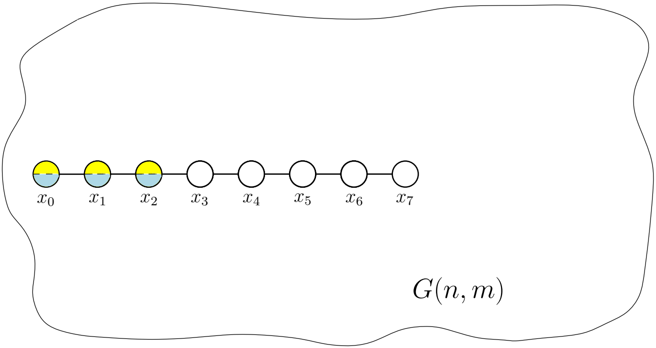

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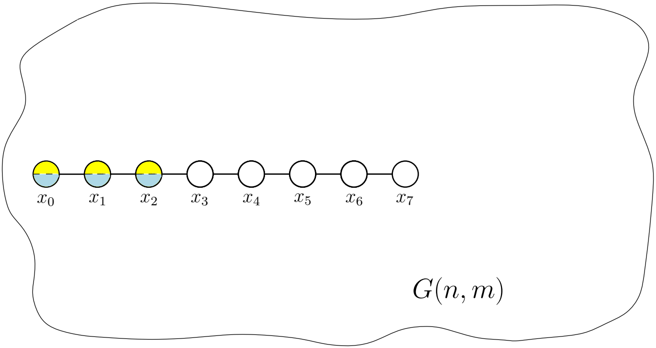

Some problems

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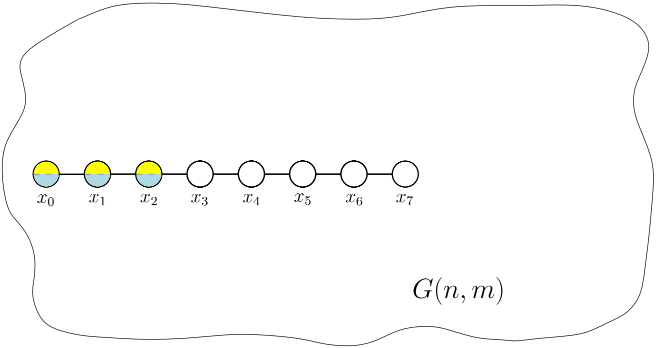

Contiguity to the rescue ...

The planted model

The planted model

Idea . . .

Reconsider the order of randomness

The planted model

Uniform Model

1. random graph $G(n, m)$
2. randomness of $\boldsymbol{\sigma}$
3. choices of `Update`

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$\boldsymbol{\sigma}^*$ is a random q-partition of the vertex set, $\ldots q = |\mathcal{S}|$

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$\boldsymbol{\sigma}^*$ is a random q-partition of the vertex set, ... $q = |\mathcal{S}|$

- the distribution of $\boldsymbol{\sigma}^*$ is **very simple**

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G^* is a **weighted** random graph on n vertices, m edges

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G^* is a **weighted** random graph on n vertices, m edges

- the weights depends on Gibbs distribution
- G^* depends on $\boldsymbol{\sigma}^*$

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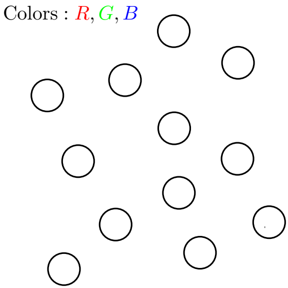
Planting Colourings



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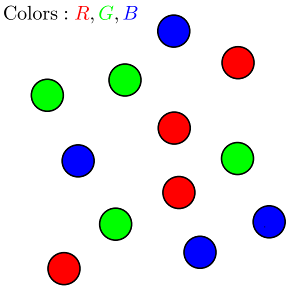

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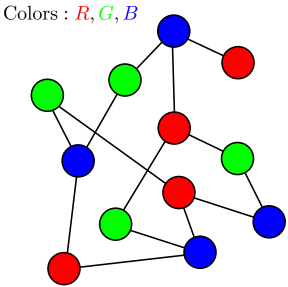

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Also known ...

in network inference they call it **Stochastic Block Model**

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- ... argue that this implies the same for the "real process"

Contiguity

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Planted pair $(\boldsymbol{G}^*, \boldsymbol{\sigma}^*)$

- $\boldsymbol{\sigma}^*$ a $|\mathcal{S}|$-partion of vertices
- generate $\boldsymbol{G}^* = \boldsymbol{G}^*(\boldsymbol{\sigma}^*)$

Contiguity

Contiguity

Definition

We say that $(\boldsymbol{G}, \boldsymbol{\sigma})$ and $(\boldsymbol{G}^*, \boldsymbol{\sigma}^*)$ are **mutual contiguous** when for any property $\mathcal{A}_n$ we have that

$$\lim_{n\to\infty} \Pr[(\boldsymbol{G}^*, \boldsymbol{\sigma}^*) \in \mathcal{A}_n] = 0 \quad \text{iff} \quad \lim_{n\to\infty} \Pr[(\boldsymbol{G}, \boldsymbol{\sigma}) \in \mathcal{A}_n] = 0.$$

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Contiguity implies ...

the two distributions have the same typical properties

Back to sampling

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Update for Teacher-Student

the input of the process is the pair $(\boldsymbol{G}^*, \boldsymbol{\sigma}^*)$

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- . . . the same holds for the "real process"

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- ... the same holds for the "real process"
- the above is **true due to contiguity**

Overview of the result

High level

We proved that

For **symmetric** Gibbs distributions on $G(n, m)$ that

1. parameters in the (conjectured) Gibbs uniqueness
   - parametrised w.r.t. the expected degree $d > 0$
2. exhibit contiguity with the corresponding teacher-student model

there is an $O(n^2 \log n)$ time sampler such that the following holds: with probability $1 - o(1)$ over $G(n, m)$ the output error is $n^{-\Omega(1)}$.

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Uniqueness Vs Contiguity

contiguity is much weaker a notion than uniqueness

Results

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Which parameters?

(conjectured) tree uniqueness parametrised w.r.t. the expected degree d

Results

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Contiguity

- Coja-Oghlan, Krzakala, Perkins, Zdeborova 2017
- Coja-Oghlan, Efthymiou, Jaafari, Kang, Kapetanopoulos 2017
- Coja-Oghlan, Kapetanopoulos, Muller, 2018

For exact statement of results . . .

On sampling symmetric Gibbs distributions on sparse random graphs and hypergraphs
https://arxiv.org/abs/2007.07145

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  - broadcasting models and probabilities

The end

Thank you!