# Inference and Mutual Information on Random Factor Graphs

Matija Pasch
LMU Munich

with A. Coja-Oghlan, M. Hahn-Klimroth, P. Loick (Goethe Frankfurt)
N. Müller, K. Panagiotou (LMU Munich)

*Inference Problems: Algorithms and Lower Bounds*
*Workshop at the Goethe University Frankfurt on September 4th, 2020*

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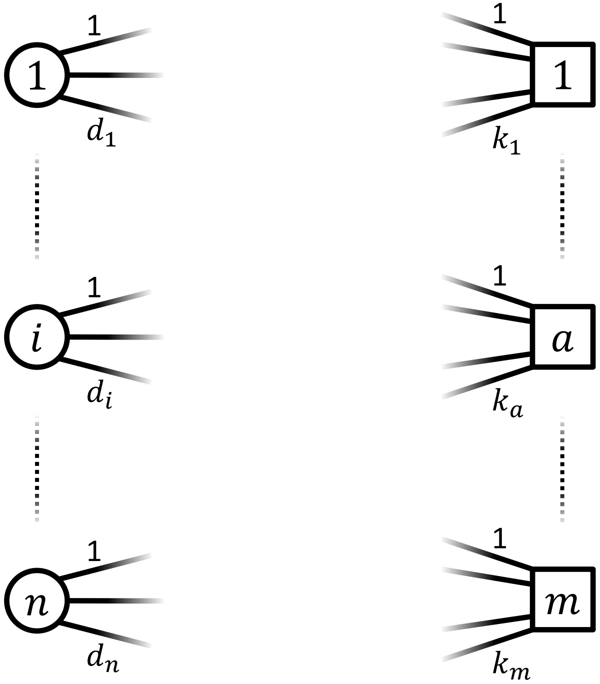
# Random Factor Graphs

- **Model Parameters**
  - Random variable degree $\boldsymbol{d} \geq 0$
  - Random factor degree $\boldsymbol{k} \geq 0$
  - Number $q \geq 2$ of colours
  - Random weights $\boldsymbol{\psi}_k : [q]^k \to (0, \infty)$ per factor degree $k$

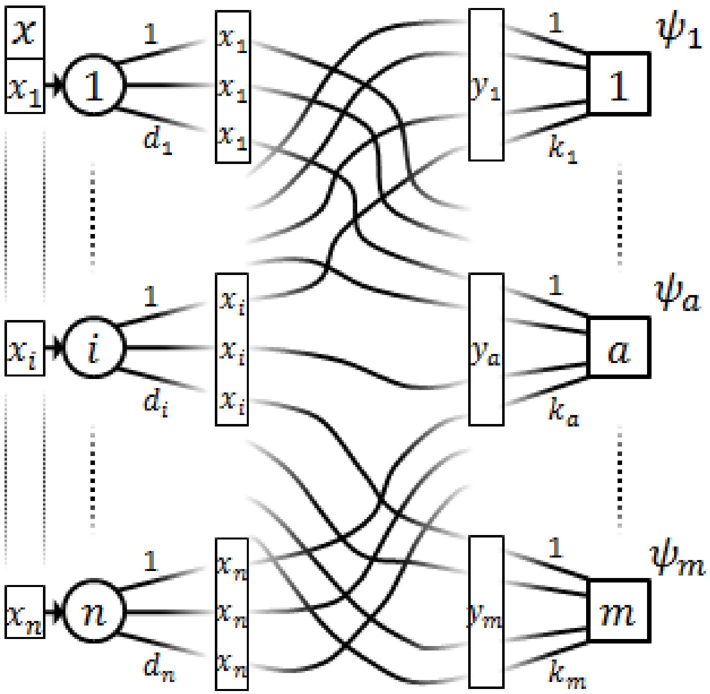



$$\sum_{i=1}^{n} d_i \quad = \quad \sum_{a=1}^{m} k_a$$

- **Degree Sequences**
  - Number $n > 0$ of variables
  - Number $\boldsymbol{m} = \mathrm{Po}(\overline{m})$ of factors $\overline{m} = \mathbb{E}[\boldsymbol{d}]n / \mathbb{E}[\boldsymbol{k}]$
  - I.i.d. degrees $\boldsymbol{d}_1, \dots, \boldsymbol{d}_n$ from $\boldsymbol{d}$ and $\boldsymbol{k}_1, \dots, \boldsymbol{k}_{\boldsymbol{m}}$ from $\boldsymbol{k}$
  - Draw $\boldsymbol{t}$ given by $(\boldsymbol{m}, \boldsymbol{d}_1, \dots, \boldsymbol{d}_n, \boldsymbol{k}_1, \dots, \boldsymbol{k}_{\boldsymbol{m}}) \mid \sum_{i=1}^{n} \boldsymbol{d}_i = \sum_{a=1}^{\boldsymbol{m}} \boldsymbol{k}_a$

# Random Factor Graphs

- **Null Model**
  - Fix factor count and degrees $t = (m, d_1, \dots, d_n, k_1, \dots, k_m)$
  - Uniform bipartite (multi-)graph
  - Weights $\boldsymbol{\psi}_a : [q]^{k_a} \to (0, \infty)$ from $\boldsymbol{\psi}_{k_a}$ for $1 \le a \le m$ independently
  - Null model $\boldsymbol{G}_t$ for fixed $t$ is the resulting random factor graph
  - Null model $\boldsymbol{G} = \boldsymbol{G}_{\boldsymbol{t}}$ over the random degree sequences $\boldsymbol{t}$

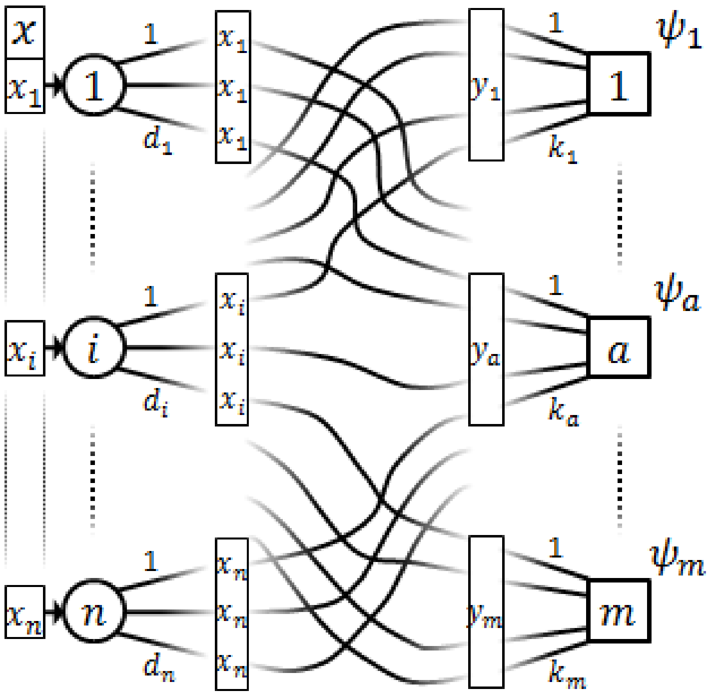



$$\sum_{i=1}^{n} d_i \quad = \quad \sum_{a=1}^{m} k_a$$

- **Assignments**
  - Assignment $x \in [q]^n$ maps to assignment $y = (y_1, \dots, y_m)$ via $G$
  - Weight of $x$ with respect to $G$ is $\psi_G(x) = \prod_{a=1}^{m} \psi_a(y_a)$

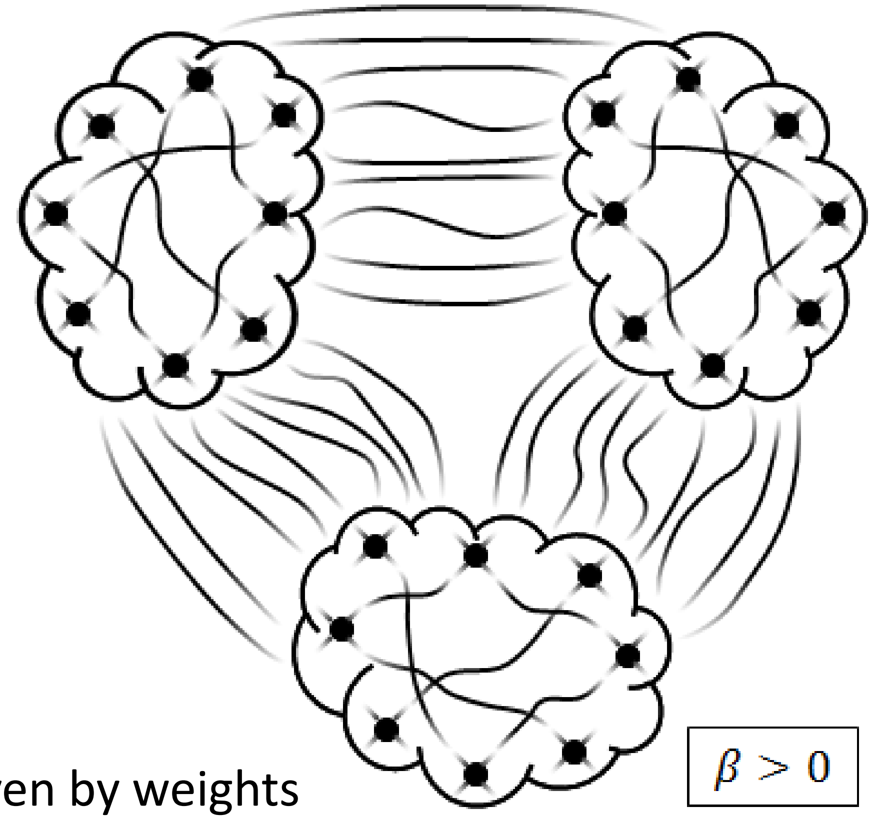
# Random Factor Graphs

- **Teacher-Student Model**
  - Fixed degrees $t$
  - Fixed ground truth $x \in [q]^n$
  - Teacher-Student Model $\boldsymbol{G}_t^*(x)$ has Radon-Nikodym derivative $G \mapsto \psi_G(x)/\mathbb{E}[\psi_{\boldsymbol{G}_t}(x)]$ with respect to $\boldsymbol{G}_t$
  - Uniform ground truth $\boldsymbol{x}^* \in [q]^n$
  - Teacher-Student Model $\boldsymbol{G}^* = \boldsymbol{G}_t^*(\boldsymbol{x}^*)$ over random ground truth and degrees




$$\sum_{i=1}^{n} d_i = \sum_{a=1}^{m} k_a$$

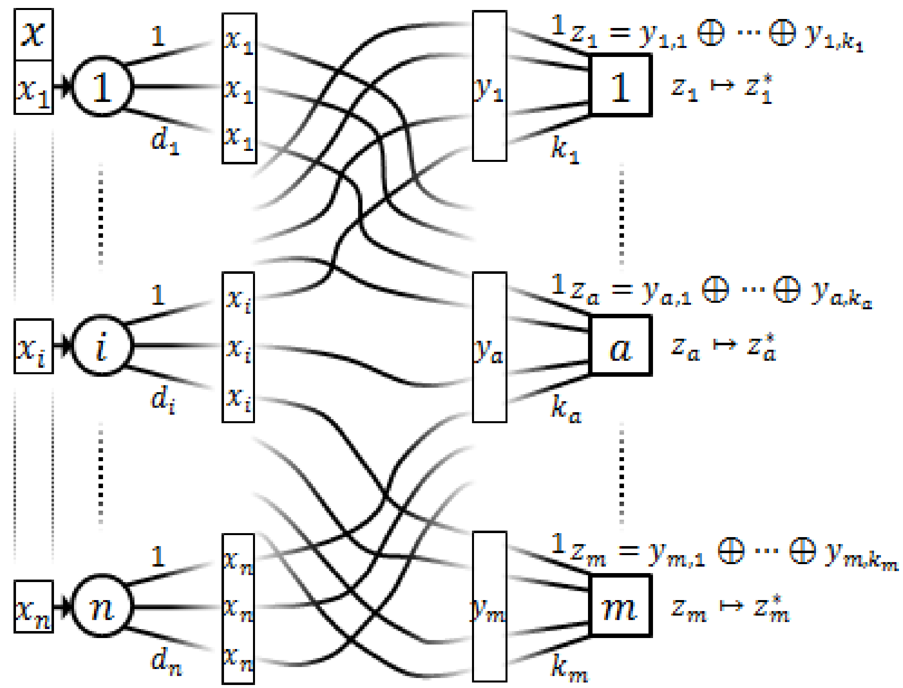
# Contents




- **Model Parameters (Regular)**
  - Number $q \geq 2$ of communities
  - Vertex degree $d \geq 3$
  - Penalty/inverse temperature $\beta$
- **Null Model**
  - Number $n > 0$ of vertices
  - Uniform $d$-regular graph $\boldsymbol{G}$
- **Stochastic Block Model**
  - Partition $x \in [q]^n$ of vertices
  - Stochastic block model $\boldsymbol{G}^*(x)$ given by weights $\exp\left(-\beta \sum_{ij \in E(G)} \mathbb{1}\{x_i = x_j\}\right) = \prod_{ij} \exp\left(-\beta \mathbb{1}\{x_i = x_j\}\right)$ with $G$ $d$-regular
  - Stochastic block model $\boldsymbol{G}^*$ over uniformly random partition $\boldsymbol{x}^*$
  - Parameters $\boldsymbol{d} = d$, $\boldsymbol{k} = 2$, $\boldsymbol{\psi}_2 = \psi$ a.s. with $\psi(y) = \exp(-\beta \mathbb{1}\{y_1 = y_2\})$

# Examples – Low Density Generator Matrix Codes

- **Model Parameters**
  - Random column sum $\boldsymbol{d} \geq 0$
  - Random row sum $\boldsymbol{k} \geq 0$
  - Flip probability $\eta \in (0,1/2)$




- **LDGM Codes**
  - Number $n, m$ of input/output bits
  - Degrees $d_i, k_a$ and graph $G$ with biadjacency matrix $A(G) \in \mathbb{F}_2^{m \times n}$
  - For input $x \in \mathbb{F}_2^n$ compute output $z = A(G)x \in \mathbb{F}_2^m$ and $\boldsymbol{z}^*(z) \in \mathbb{F}_2^m$ by flipping each bit with probability $\eta$
  - Output $\boldsymbol{z}^*$ for uniform graph, input $\boldsymbol{x}^*$, random degrees $\boldsymbol{t}$ (as before)
  - Parameters $q = 2$ over $\{-1,1\}$, $\boldsymbol{d}$ , $\boldsymbol{k}$, and $\boldsymbol{\psi}_k \in \{\psi_{k,-1}, \psi_{k,1}\}$ uniformly with $\psi_{k,J}(y) = 1 + (1-2\eta)J \prod_{h=1}^{k} y_h$ for $y \in \{-1,1\}^k$ and $J \in \{-1,1\}$

# Contents

- Random Factor Graphs
- Examples: Stochastic Block Model and LDGM Codes
- Inference and Mutual Information
- Proof Overview

# Inference and Mutual Information

- **Assumptions**
  - Degrees satisfy $\mathbb{E}[\boldsymbol{d}^{2+\varepsilon}], \mathbb{E}[\boldsymbol{k}^{2+\varepsilon}] \in \mathbb{R}_{>0}$
  - Support of $\boldsymbol{\psi}_k$ is finite for all $k$
    (e.g. violated by mixed $\boldsymbol{k}$-spin model, rectifiable via discretization?)
  - There exists $\varepsilon$ with $\varepsilon < \boldsymbol{\psi}_k < 1/\varepsilon$ a.s. for all $k$
    (e.g. violated by mixed $\boldsymbol{k}$-spin model, CSPs, rectifiable via capping?)
  - There exists $\xi$ with $\sum_y \mathbb{1}\{y_h = z\}\boldsymbol{\psi}_k(y) = \xi q^{k-1}$ a.s.
    for all $z \in [q]$, $h \in [k]$, $k$
    (e.g. violated by positive temperature $k$-SAT, unbalanced problems)
  - For all $k$ the map $\mathcal{P}([q]) \to \mathbb{R}_{\geq 0}, p \mapsto \sum_{y\in[q]^k} \mathbb{E}[\boldsymbol{\psi}_k(y)] \prod_{h\in[k]} p(y_h)$,
    is concave and maximal at the uniform distribution $u_{[q]} \in \mathcal{P}([q])$
    (violated by unbalanced problems, e.g. perfect matchings in hypergraphs)
  - Convexity assumption **POS** …
    (e.g. violated by the assortative stochastic block model, $\beta < 0$)

# Inference and Mutual Information

- **Mutual Information**
  - Fix $n$ and $t = (m, d_1, \ldots, d_n, k_1, \ldots, k_m)$
  - Mutual information per variable given degrees $t$
    $i(t) = {}^1/_n\, I(\boldsymbol{G}_t^*(\boldsymbol{x}^*), \boldsymbol{x}^*)$
    $= {}^1/_n \sum_{G,x} \mathbb{P}[\boldsymbol{G}_t^*(x) = G, \boldsymbol{x}^* = x] \ln \frac{\mathbb{P}[\boldsymbol{G}_t^*(x) = G, \boldsymbol{x}^* = x]}{\mathbb{P}[\boldsymbol{G}_t^*(x) = G]\mathbb{P}[\boldsymbol{x}^* = x]}$
  - Mutual information per variable
    $i_n = {}^1/_n\, I(\boldsymbol{G}^*, \boldsymbol{x}^*) = \mathbb{E}[i(\boldsymbol{t})]$

# Inference and Mutual Information

- **Bethe Functional**
  - Set $\mathcal{P}_*^2([q]) \subseteq \mathcal{P}\big(\mathcal{P}([q])\big)$ of distributions $\pi$ over $\mathcal{P}([q]) \subseteq \mathbb{R}^q$ with $\mathbb{E}[\boldsymbol{\mu}] = u_{[q]}$ uniform, where $\boldsymbol{\mu} \in \mathcal{P}([q])$ has law $\pi$
  - Fix $q$, $\boldsymbol{d}$, $\boldsymbol{k}$, $\boldsymbol{\psi}_k$ satisfying the assumptions, let $\xi$ with $\sum_y \mathbb{1}\{y_h = z\}\boldsymbol{\psi}_k(y) = \xi q^{k-1}$ a.s. and $\pi \in \mathcal{P}_*^2([q])$
  - Reweighted degree $\mathbb{P}\big[\hat{\boldsymbol{k}} = k\big] = k\mathbb{P}[\boldsymbol{k} = k]/\mathbb{E}[\boldsymbol{k}]$, i.i.d. copies $\hat{\boldsymbol{k}}_1, \ldots$
  - I.i.d. copies $\boldsymbol{\mu}_{a,h}$ and $\boldsymbol{\mu}_h$ with law $\pi$ for $a, h \geq 1$
  - I.i.d. copies $\boldsymbol{\psi}_{k,a}$ of $\boldsymbol{\psi}_k$ for $a \geq 1$
  - Independent uniform $\boldsymbol{h}_{k,a} \in [k]$ for $a \geq 1$ and all $k$
  - Using $\Lambda(x) = x\ln(x)$ the Bethe functional is given by

$$\mathcal{B}(\pi) = \frac{1}{q}\mathbb{E}\left[\frac{1}{\xi^{\boldsymbol{d}}}\Lambda\left(\sum_{x=1}^{q}\prod_{a=1}^{\boldsymbol{d}}\sum_{y\in[q]^{\hat{\boldsymbol{k}}_a}}\mathbb{1}\left\{y_{\boldsymbol{h}_{\hat{\boldsymbol{k}}_a,a}} = x\right\}\boldsymbol{\psi}_{\hat{\boldsymbol{k}}_a,a}(y)\prod_{h\in[\hat{\boldsymbol{k}}_a]\setminus\left\{\boldsymbol{h}_{\hat{\boldsymbol{k}}_a,a}\right\}}\boldsymbol{\mu}_{a,h}(y_h)\right)\right]$$
$$-\frac{\mathbb{E}[\boldsymbol{d}]}{\xi\mathbb{E}[\boldsymbol{k}]}\mathbb{E}\left[(\boldsymbol{k}-1)\Lambda\left(\sum_{y\in[q]^{\boldsymbol{k}}}\boldsymbol{\psi}_{\boldsymbol{k}}(y)\prod_{h\in[\boldsymbol{k}]}\boldsymbol{\mu}_h(y_h)\right)\right]$$

# Inference and Mutual Information

- **Free Entropy Density**
  - Partition function $Z_G = \sum_{x \in [q]^n} \psi_G(x)$ for factor graph $G$
  - Free entropy density $\phi_G = \frac{1}{n} \ln(Z_G)$
  - Average $\phi^*(t) = \mathbb{E}\left[\phi_{\boldsymbol{G}_t^*(\boldsymbol{x}^*)}\right]$ for fixed degrees $t$

**Proposition** (Quenched Free Entropy Density TSS)

We have $\lim_{n\to\infty} \mathbb{E}[\phi^*(\boldsymbol{t})] = \lim_{n\to\infty} \mathbb{E}[\phi_{\boldsymbol{G}^*}] = \sup_{\pi \in \mathcal{P}_*^2([q])} \mathcal{B}(\pi)$.

**Proposition** (Quenched Free Entropy Density TSS)

$\phi^*(\boldsymbol{t})$ converges to $\sup_{\pi \in \mathcal{P}_*^2([q])} \mathcal{B}(\pi)$ in probability.

# Inference and Mutual Information

- **Mutual Information Asymptotics**
  - Mutual information $i(t)$ boils down to $\phi^*(t)$ for typical degrees $t$
  - We recover the limit of the expectation $i_n$ and in probability
  - Using $\Lambda(x) = x\ln x$ and $\xi$ from the assumptions

**Theorem** (Mutual Information)

$$\lim_{n\to\infty} i_n = \ln(q) + \frac{\mathbb{E}[\boldsymbol{d}]}{\xi\mathbb{E}[\boldsymbol{k}]}\mathbb{E}\left[\frac{1}{q^{\boldsymbol{k}}}\sum_{y\in[q]^{\boldsymbol{k}}}\Lambda(\boldsymbol{\psi}_{\boldsymbol{k}}(y))\right] - \sup_{\pi\in\mathcal{P}_*^2([q])}\mathcal{B}(\pi)$$

**Theorem** (Mutual Information)

$i(\boldsymbol{t})$ converges to $\lim_{n\to\infty} i_n$ in probability.

# Inference and Mutual Information

- **Condensation Regime**
  - Annealed free entropy density $\phi_{\mathrm{a}} = \lim_{n\to\infty} \mathbb{E}[\phi_{\mathrm{a},\boldsymbol{t}}]$ with $\phi_{\mathrm{a},t} = \frac{1}{n}\ln(\mathbb{E}[Z_{\boldsymbol{G}_t}])$ for given $n$ and degrees $t$
  - Using $\zeta_k = \sum_{y\in[q]^k} \mathbb{E}[\boldsymbol{\psi}_k(y)] = q^k\xi$ we have $\phi_{\mathrm{a}} = (1-\mathbb{E}[\boldsymbol{d}])\ln(q) + \frac{\mathbb{E}[\boldsymbol{d}]}{\mathbb{E}[\boldsymbol{k}]}\mathbb{E}[\ln(\zeta_{\boldsymbol{k}})] = \ln(q) + \frac{\mathbb{E}[\boldsymbol{d}]}{\mathbb{E}[\boldsymbol{k}]}\ln(\xi)$
  - With $q$ fixed, $\phi_{\mathrm{a}}$ and $\mathcal{B} = \sup_{\pi\in\mathcal{P}_*^2([q])}\mathcal{B}(\pi)$ depend on $\boldsymbol{d}$, $\boldsymbol{k}$ and $(\boldsymbol{\psi}_k)_k$
  - Replica symmetric regime $\mathcal{R}_{\mathrm{rs}} = \{(\boldsymbol{d},\boldsymbol{k},(\boldsymbol{\psi}_k)_k) : \mathcal{B} \le \phi_{\mathrm{a}}\}$ (actually $\mathcal{B} = \phi_{\mathrm{a}}$)
  - Condensation regime $\mathcal{R}_{\mathrm{cond}} = \{(\boldsymbol{d},\boldsymbol{k},(\boldsymbol{\psi}_k)_k) : \mathcal{B} > \phi_{\mathrm{a}}\}$
  - Canonical generalization of the condensation threshold for the binomial model

# Inference and Mutual Information

- **Condensation Results**
  - Boltzmann distribution $\mathbb{P}[\boldsymbol{x}_G = x] = \psi_G(x)/Z_G$ for factor graph $G$
  - Relative entropy per variable of models
    $d_n = \frac{1}{n} D(\boldsymbol{x}^*, \boldsymbol{G}^* || \boldsymbol{x}_G, \boldsymbol{G}) = \frac{1}{n} \sum_{x,G} \mathbb{P}[\boldsymbol{x}^* = x, \boldsymbol{G}^* = G] \ln \frac{\mathbb{P}[\boldsymbol{x}^* = x, \boldsymbol{G}^* = G]}{\mathbb{P}[\boldsymbol{x}_G = x, \boldsymbol{G} = G]}$

**Theorem** (Quenched Free Entropy Density)

$\lim_{n\to\infty} \mathbb{E}[\phi_{\boldsymbol{G}}] = \phi_{\mathrm{a}}$ (in $\mathcal{R}_{\mathrm{rs}}$) and $\limsup_{n\to\infty} \mathbb{E}[\phi_{\boldsymbol{G}}] < \phi_{\mathrm{a}}$ (in $\mathcal{R}_{\mathrm{cond}}$)

**Theorem** (Relative Entropy)

$\lim_{n\to\infty} d_n = 0$ (in $\mathcal{R}_{\mathrm{rs}}$) and $\liminf_{n\to\infty} d_n > 0$ (in $\mathcal{R}_{\mathrm{cond}}$)

# Inference and Mutual Information

- **LDGM Codes**
  - Can the input $x$ be recovered from the scrambled output $z^*$?
  - Conjecture by Montanari (2005) confirmed (for the standard ensemble)
  - Results by Coja-Oghlan et al. (2018) and van den Brand, Jaafari (2017) significantly extended
- **Stochastic Block Model**
  - Can the communities $x \in [q]^n$ be recovered from the graph $G$?
  - Threshold $\beta^*$ for the $d$-regular disassortative case is infimum of condensation regime (in $\beta$)
  - For $\beta > \beta^*$ there exists an algorithm that approximates $x$
- **Other Models**
  - Long-range correlations for the mixed $\boldsymbol{k}$-spin model
  - …

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# Proof Overview

- **Mutual Information and Free Entropy**
  - Configuration model with variable/factor clones and bijections
  - Factor assignment $\boldsymbol{y}_t(x)$ for ground truth $x$ under $\boldsymbol{G}_t^*(x)$ and derived model $\boldsymbol{G}_t^*(x, y)$ given the factor assignment $y$ induced by $x$
  - Nishimori model $\hat{\boldsymbol{x}}_t \in [q]^n$ with weights $\mathbb{E}\left[\psi_{\boldsymbol{G}_t}(x)\right] / \mathbb{E}\left[Z_{\boldsymbol{G}_t}\right]$ and $\widehat{\boldsymbol{G}}_t$ with RN derivative $Z_G / \mathbb{E}\left[Z_{\boldsymbol{G}_t}\right]$ wrt $\boldsymbol{G}_t$ for fixed degrees $t$
  - Nishimori identity: $(\hat{\boldsymbol{x}}_t, \boldsymbol{G}_t^*(\hat{\boldsymbol{x}}_t))$ and $\left(\boldsymbol{x}_{\widehat{\boldsymbol{G}}_t}, \widehat{\boldsymbol{G}}_t\right)$ have the same law
  - Typical degrees $t$: asymptotical properties and uniform bounds
  - Mutual contiguity of $\hat{\boldsymbol{x}}_t$ and $\boldsymbol{x}^*$: limit distributions of colour frequencies (point probability asymptotics in large deviation regime, uniform bounds)
  - Concentration of variable/factor assignment frequencies (given typical degrees and colour frequencies with uniform bounds)
  - Determine asymptotics of the mutual information $i(t)$ per variable up to the expected free entropy density $\phi^*(t)$

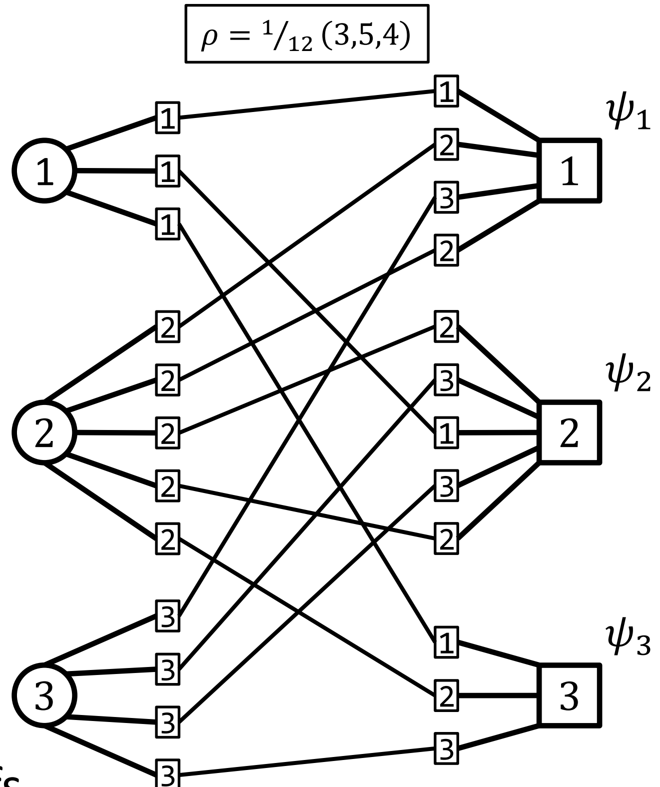
# Proof Overview

- **Free Entropy and Degree Capping**
  - Free entropy concentration for degrees $t$ and assignments $x$, $y$ around $\phi^*(t,x,y) = \mathbb{E}\left[\phi_{\boldsymbol{G}_t^*(x,y)}\right]$ using Azuma gives local concentration
  - Lipschitz continuity of $\phi^*(t,x,y)$ using degree/assignment concentration gives global concentration with $\phi^* \approx \phi^*(t) \approx \phi^*(t,x,y)$
  - Bounds on the distance of $\phi^*(t,x,y)$ and $\phi^*(t',x',y')$ in terms of the distance of $(\boldsymbol{d},\boldsymbol{k})$ and $(\boldsymbol{d}',\boldsymbol{k}')$ for typical outcomes $(t,x,y)$, $(t',x',y')$
  - Uniform continuity of the Bethe functional wrt $(\boldsymbol{d},\boldsymbol{k})$
  - Proving $\lim_{n\to\infty}\mathbb{E}[\phi_{\boldsymbol{G}^*}] = \sup_{\pi\in\mathcal{P}_*^2([q])}\mathcal{B}(\pi)$ for finitely supported $\boldsymbol{d}$, $\boldsymbol{k}$ is sufficient to obtain the general case

# Proof Overview

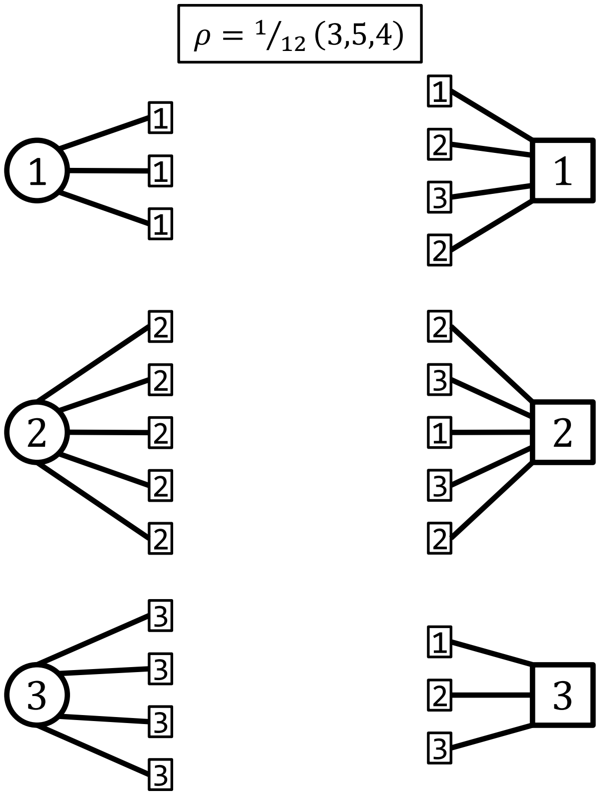
- **Free Entropy and the Bethe Functional**
  - Cavity model with $\mathrm{Po}((1-\varepsilon)\overline{m})$ distributed number $\boldsymbol{m}_\varepsilon$ of factors and degrees $\boldsymbol{t}_\varepsilon$ subject to $\sum_{i=1}^{n} \boldsymbol{d}_i \geq \sum_{a=1}^{\boldsymbol{m}_\varepsilon} \boldsymbol{k}_a$, resulting in unmatched variable clones (cavities)
  - Variable Pinning: Randomly choose a few variables and fix their value to the ground truth assignment (using factors)
  - Sufficient wiggle room for couplings and control over the dependencies of the coordinates of the posterior $\boldsymbol{x}_{\boldsymbol{G}^*}$
  - Aizenman-Sims-Starr scheme yields $\lim_{n\to\infty} \mathbb{E}[\phi_{\boldsymbol{G}^*}] \leq \sup_{\pi\in\mathcal{P}_*^2([q])} \mathcal{B}(\pi)$
  - Interpolation method yields $\lim_{n\to\infty} \mathbb{E}[\phi_{\boldsymbol{G}^*}] \geq \sup_{\pi\in\mathcal{P}_*^2([q])} \mathcal{B}(\pi)$

# Proof Overview




- **Teacher-Student Model**
  - Fix degrees $t = (m, d_1, \ldots, d_n, k_1, \ldots, k_m)$
  - Fix consistent assignments $x, y$ with same colour frequencies $\rho$ on clones
  - *Independent* bijections $\boldsymbol{g}_z$ for $z \in [q]$
  - *Independent* weights per factor $a$
    RN derivative $\psi \mapsto \psi(y_a)/\mathbb{E}[\boldsymbol{\psi}_{k_a}(y_a)]$
    with respect to $\boldsymbol{\psi}_{k_a}$
  - Resulting factor graph is $\boldsymbol{G}_t^*(x, y)$
  - Facilitates coupling and concentration proofs
  - Requires discussion of colour frequencies under $\boldsymbol{x}^*$ and $\widehat{\boldsymbol{x}}_t$
    → Convergence to normal centered at $u_{[q]}$ in both cases
  - Requires discussion of assignment frequencies under $(\boldsymbol{x}^*, \boldsymbol{y}_t(\boldsymbol{x}^*))$ and $(\widehat{\boldsymbol{x}}_t, \boldsymbol{y}_t(\widehat{\boldsymbol{x}}_t))$ (given colour frequencies)
  - Canonically translates to cavity model

# Proof Overview




- **Ground Truth Assignments**
  - Fix degrees $t = (m, d_1, \ldots, d_n, k_1, \ldots, k_m)$
  - Fix (valid) colour frequencies $\rho \in \mathcal{P}([q])$
    Controlled by LLT (for both $\boldsymbol{x}^*$ and $\hat{\boldsymbol{x}}_t$)
  - Assignments $\boldsymbol{x}^*_{t,\rho}$ and $\boldsymbol{y}^*_{t,\rho} = \boldsymbol{y}_t\left(\boldsymbol{x}^*_{t,\rho}\right)$
    given $t$ and $\rho$ are *independent*
    So are the Nishimori versions $\hat{\boldsymbol{x}}_{t,\rho}$ and $\hat{\boldsymbol{y}}_{t,\rho}$
  - $\boldsymbol{x}^*_{t,\rho}$ and $\hat{\boldsymbol{x}}_{t,\rho}$ have the same law
  - Counterpart $\boldsymbol{y}_t = \left(\boldsymbol{y}_{t,1}, \ldots, \boldsymbol{y}_{t,m}\right)$ to $\boldsymbol{x}^*$
    given by $\mathbb{P}\left[\boldsymbol{y}_{t,a} = y\right] = \frac{\mathbb{E}[\boldsymbol{\psi}_{k_a}(y)]}{\sum_{y'} \mathbb{E}[\boldsymbol{\psi}_{k_a}(y')]}$ for each factor $a$ *independently*
  - Colour frequencies of $\boldsymbol{y}_t$ concentrate around $u_{[q]}$ since $\boldsymbol{\psi}_k$'s are balanced
  - Both $\boldsymbol{y}^*_{t,\rho}$ and $\hat{\boldsymbol{y}}_{t,\rho}$ have the same law as $\boldsymbol{y}_{t,\rho}$
  - Canonically translates to cavity model

# Proof Overview

- **Conclusion**
  - Control $\boldsymbol{t}$, fix degree sequences $t = (m, d_1, \dots, d_n, k_1, \dots, k_m)$
  - Control $\boldsymbol{\rho}_t^*$ and $\widehat{\boldsymbol{\rho}}_t$, fix colour frequencies $\rho$
  - Control assignments $\boldsymbol{x}_{t,\rho}^*$ and $\boldsymbol{y}_{t,\rho}$, fix $x$ and $y$
  - Use independence of the components of $\boldsymbol{G}_t^*(x, y)$ to control the free entropy density, mutual information per variable, …

- **Next Steps**
  - Weaken/remove assumptions used for convenience
  - Work towards zero temperature limit
  - Consider unbalanced problems
  - Strengthen connections to RSB theory

# Thank you!

# References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